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Singular unitarity in "quantization commutes with reduction"

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Abstract

Let M be a connected compact quantizable Kähler manifold equipped with a Hamiltonian action of a connected compact Lie group G. Let $M//G = \phi^{-1}(0)/G = M_0$ be the symplectic quotient at value 0 of the moment map ϕ. The space M_0 may in general not be smooth. It is known that, as vector spaces, there is a natural isomorphism between the quantum Hilbert space over M_0 and the G-invariant subspace of the quantum Hilbert space over M. In this paper, without any regularity assumption on the quotient M_0, we discuss the relation between the inner products of these two quantum Hilbert spaces under the above natural isomorphism; we establish asymptotic unitarity to leading order in Planck's constant of a modified map of the above isomorphism under a "metaplectic correction" of the two quantum Hilbert spaces.

MSC: 53D50; 53D20; 81S10

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1. Introduction

Let M be an integral connected compact Kähler manifold with symplectic form ω. Then M is quantizable, i.e., there is a Hermitian holomorphic line bundle L over M with a connection whose curvature is $-i\omega$. We consider the kth tensor power $L^{\otimes k}$ of L. The Hermitian structure on L induces a Hermitian structure on $L^{\otimes k}$. The Hermitian structure on $L^{\otimes k}$ naturally equips the space of holomorphic sections of $L^{\otimes k}$ over M with an inner product. For each k, the quantum Hilbert space $\mathcal{H}(M, L^{\otimes k})$ is the space of holomorphic sections of $L^{\otimes k}$ over M with the inner product.

Now, let G be a connected compact Lie group acting on M holomorphically and in Hamiltonian fashion with equivariant moment map ϕ. Let $M//G = \phi^{-1}(0)/G = M_0$ be the **reduced space** at value 0.

Let us first consider the case when the action of G on $\phi^{-1}(0)$ is free. Then M_0 is a smooth connected compact Kähler manifold. Assume that the G action lifts to L, preserving the Hermitian metric. The Hermitian line bundle $L^{\otimes k}$ naturally descends to a Hermitian line bundle $(L^{\otimes k})_0 = (L^{\otimes k}|_{\phi^{-1}(0)})/G$ over M_0. The Hermitian structure on $(L^{\otimes k})_0$ naturally equips the space of holomorphic sections of $(L^{\otimes k})_0$ over M_0 with an inner product. For each k, the quantum Hilbert space $\mathcal{H}(M_0, (L^{\otimes k})_0)$ is the space of holomorphic sections of $(L^{\otimes k})_0$ over M_0 with the inner product. This is the first "reducing" and then "quantizing" Hilbert space. The first "quantizing" and then "reducing"

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quantum Hilbert space is the G-invariant subspace $\mathcal{H}(M, L^{\otimes k})^G$ of $\mathcal{H}(M, L^{\otimes k})$. By Guillemin and Sternberg [6], there is a natural invertible linear map A_k between $\mathcal{H}(M, L^{\otimes k})^G$ and $\mathcal{H}(M_0, (L^{\otimes k})_0)$. Let us call this linear map the Guillemin–Sternberg map. For quantum mechanics, the inner products of the quantum Hilbert spaces are also important. A few authors have observed that the Guillemin–Sternberg map is not unitary, and, it does not become asymptotically unitary as $k \to \infty$. Moreover, they identified the volume of the G-orbits in the zero level set as an obstruction to asymptotic unitarity. We refer to the work of Flude [5], Paoletti [14], Ma–Zhang [12,13], Charles [3] and Hall–Kirwin [9]. Flude was the first who gave a formal computation of the leading-order term of the asymptotic density function (the function which relates the norm of an invariant holomorphic section upstairs and the norm of the descended section downstairs) and who obtained the non-unitarity result. Paoletti proved this result in his study of the asymptotic expansion of the Szegö kernels upstairs and downstairs (using microlocal analysis [1,2]). Ma and Zhang obtained this result (Theorem 0.10 for $E = \mathbb{C}$ in [13]) on their way of studying the asymptotic expansion of the G-invariant Bergman kernel of the spin^c Dirac operator associated with vector bundles on a symplectic manifold. Charles obtained this result in his study of (invariant) Toeplitz operators on M and on the symplectic quotient M_0 (for torus actions) by looking at the relations he obtained of the (principal) symbols of the Toeplitz operators on M and of the Toeplitz operators on M_0. In the recent study of the inner products of quantum Hilbert spaces by Hall and Kirwin [9], they proved again the non-unitarity result by writing down an exact expression for the norm of an invariant holomorphic section upstairs as an integral over M_0 and by estimating the leading term of the asymptotic behavior of the density function. Moreover, for this "free action" case, they obtained asymptotic unitarity results for a modified quantization procedure. More precisely, they took the tensor products of the line bundles $L^{\otimes k}$'s with the square root of the canonical bundle of M (assuming it exists), called the metaplectic correction, and they showed that a new defined Guillemin–Sternberg type map B_k between the new quantum Hilbert spaces is invertible for all sufficiently large k, and that this map is asymptotically unitary to leading order as $k \to \infty$.

In general, the action of G on $\phi^{-1}(0)$ may not be free. Consequently, the quotient M_0 may not be smooth. By [15] and by [16], M_0 is in general a stratified Kähler space, with the stratification being given by orbit types of the action. When there is only one orbit type, M_0 is still a smooth Kähler manifold. In this general case when the action of G on $\phi^{-1}(0)$ may not be free, the Hermitian line bundle $L^{\otimes k}$ descends to a Hermitian V-line bundle $(L^{\otimes k})_0$ over M_0. Let $\mathcal{H}(M_0, (L^{\otimes k})_0)$ still be the space of holomorphic sections of the V-line bundle $(L^{\otimes k})_0$ over M_0 with the induced inner product. By Sjamaar (see Theorem 6), there is a natural linear isomorphism A'_k between $\mathcal{H}(M, L^{\otimes k})^G$ and $\mathcal{H}(M_0, (L^{\otimes k})_0)$.

When the action of G on $\phi^{-1}(0)$ is not free, the volume of the G-orbits in $\phi^{-1}(0)$ is of course less "uniform". One guesses by the above authors' results that A'_k would not be unitary or asymptotically unitary after suitable quantum norms are defined. In this paper, we drop the assumption that the action of G on $\phi^{-1}(0)$ is free. We give a formula for the relation of the quantum norm of an invariant holomorphic section upstairs and the quantum norm of the descended section downstairs under the map A'_k, and we give an asymptotic formula of this to leading order as $k \to \infty$. We see that A'_k is not unitary and it is not asymptotically unitary. We still consider the "metaplectic correction". We give a description of how the square root of the canonical bundle of M descends to M_0, we show the existence of a family of modified isomorphisms B'_k for sufficiently large k between the new quantum Hilbert spaces, and we establish asymptotic unitarity to leading order term for the maps B'_k.

There are two main problems that need to be addressed in this new study. One is that we find a suitable way to descend the half form bundle of M to the stratified quotient M_0. Another problem has to do with a large piece of the manifold M, the semistable set M^{ss}, which is open dense and connected in M. By Theorem 6, $\mathcal{H}(M_0, (L^{\otimes k})_0) \simeq \mathcal{H}(M^{ss}, L^{\otimes k})^G$. The holomorphic action of G can be analytically extended to a $G_{\mathbb{C}}$-action, where $G_{\mathbb{C}}$ is the complexification of G. If G acts freely on $\phi^{-1}(0)$, M^{ss} consists of free G-orbits and it consists of complex $G_{\mathbb{C}}$-orbits, each of which intersects $\phi^{-1}(0)$ at one G-orbit. In the general case, M^{ss} may contain complex $G_{\mathbb{C}}$-orbits which do not intersect $\phi^{-1}(0)$ but contain those $G_{\mathbb{C}}$-orbits which intersect $\phi^{-1}(0)$ in their closures. We will analyze the structure of these complex orbits and study their contribution to the quantum norms.

Our main results are Theorems 7, 9, 11 and 12 (and the corollaries of Theorems 11 and 12, Corollaries 2 and 3), and Theorem 14.

We will use three different notations interchangeably for the symplectic quotient at 0, M_0, $M//G$, and $M^{ss}//G_{\mathbb{C}}$, depending on the context.

2. Reduction of Kähler manifolds

In this section, we will recall some main results obtained by Sjamaar in [16] on general Kähler quotients. This will help us to understand our space M_0 as well as its relation with M. One may see the difference between the case when the action of G on $\phi^{-1}(0)$ is free and the case when this assumption is removed. This section also serves as a preparation for the tools needed in the subsequent sections.

To understand stratified Kähler spaces and their quantizations, we also refer to the work of Huebschmann, [7,8].

Let $(M, \omega, J, B = \omega(\cdot, J\cdot))$ be a connected compact Kähler manifold with symplectic form ω, compatible complex structure J and Riemannian metric B. Let G be a connected compact Lie group acting holomorphically on M. Assume that the G action is Hamiltonian with an equivariant moment map ϕ. Assume a is a value of ϕ. Then the quotient $M_a = \phi^{-1}(G \cdot a)/G$ is called the symplectic quotient or the reduced space at the coadjoint orbit $G \cdot a$. Let us restrict attention to the value $a = 0$. By [15], the quotient M_0 is a connected compact stratified symplectic space with a connected open dense stratum. If M_0 has only one stratum, then it is a smooth symplectic manifold. We will see that M_0 also admits an analytic structure such that M_0 is a stratified Kähler space.

Since G acts holomorphically, the action can be analytically continued to a holomorphic action of the "complexified" group $G_{\mathbb{C}}$ on M. The Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$. The Cartan decomposition gives a diffeomorphism $G_{\mathbb{C}} \backsimeq \exp(i\mathfrak{g})G$. For $\xi \in \mathfrak{g}$, let X^{ξ} be the infinitesimal vector field on M generated by ξ. Then $X^{i\xi} = JX^{\xi}$ is the infinitesimal vector field generated by $i\xi$.

Define a point m in M to be **(analytically) semistable** if the closure of the $G_{\mathbb{C}}$-orbit through m intersects the zero level set $\phi^{-1}(0)$. Let M^{ss} be the set of semistable points in M. The point m is called **(analytically) stable** if the closure of the $G_{\mathbb{C}}$-orbit through m intersects the zero level set $\phi^{-1}(0)$ at a point where $d\phi$ is surjective. Let M^s be the set of stable points in M. When the action of G on $\phi^{-1}(0)$ is free or locally free, $d\phi$ is surjective at any point of $\phi^{-1}(0)$. In this case, M^{ss} coincides with M^s.

Assuming there is a G-invariant inner product on $\mathfrak{g}$, by Lemma 6.6 in [10], the gradient vector field of $\|\phi\|^2$ is given by

$$\mathrm{grad}(\|\phi\|^2)(m) = 2JX^{\phi(m)}(m),$$

where we have identified $\phi(m) \in \mathfrak{g}^*$ with a vector in $\mathfrak{g}$ using the inner product, and where $X^{\phi(m)}(m)$ is the vector field on M induced by $\phi(m)$, evaluated at the point m. So $\mathrm{grad}(\|\phi\|^2)(m)$ is tangent to the $G_{\mathbb{C}}$-orbits. Let F_t be the flow of $-\mathrm{grad}(\|\phi\|^2)$. Kirwan has proved that M^{ss} is the set of points $m \in M$ such that the path $F_t(m)$ has a limit point in $\phi^{-1}(0)$ ([10]). By [11] or by [17], the limit map $F_\infty(m)$ gives an equivariant deformation retraction from M^{ss} onto $\phi^{-1}(0)$.

2.1. The holomorphic slice theorem

In order to describe the complex analytic structure on M_0, let us first recall the holomorphic slice theorem due to R. Sjamaar. The results on the orbit structure of M^{ss} and on the stratified Kähler structure of M_0 are due to this theorem.

Theorem 1 (*Holomorphic Slice Theorem [16]*). *Let M be a Kähler manifold and let $G_{\mathbb{C}}$ act holomorphically on M. Assume that the action of the compact real form G is Hamiltonian. Let m be any point in M such that the G-orbit through m is isotropic. Then there exists a* **holomorphic slice** *at m for the $G_{\mathbb{C}}$-action.*

If X is a complex space and $G_{\mathbb{C}}$ a reductive complex Lie group acting holomorphically on X, we have the following definition of a **holomorphic slice**.

Definition 1. A **holomorphic slice** at x for the $G_{\mathbb{C}}$ action is a locally closed analytic subspace D of X with the following properties:

1. $x \in D$;
2. $G_{\mathbb{C}}D$ of D is open in X;
3. D is invariant under the action of the stabilizer $(G_{\mathbb{C}})_x$;
4. the natural $G_{\mathbb{C}}$-equivariant map from the associated bundle $G_{\mathbb{C}} \times_{(G_{\mathbb{C}})_x} D$ into X, which sends an equivalence class $[g, y]$ to the point gy, is an analytic isomorphism onto $G_{\mathbb{C}}D$.

2.2. Kähler reduction

For the orbit structure of M^{ss}, one may see Proposition 2.4 in [16]. We list a few of them which are more relevant to us.

Proposition 1. *In the following, "closed" means "closed in M^{ss}" and "closure" means "closure in M^{ss}".*

1. *The semistable set M^{ss} is the smallest $G_{\mathbb{C}}$-invariant open subset of M containing $\phi^{-1}(0)$, and its complement in M is a complex-analytic subset;*
2. *A $G_{\mathbb{C}}$-orbit in M^{ss} is closed if and only if it intersects $\phi^{-1}(0)$;*
3. *The closure of every $G_{\mathbb{C}}$-orbit in M^{ss} contains exactly one closed $G_{\mathbb{C}}$-orbit.*

We call two semistable points x and y **related** if the closures in M^{ss} of the orbits $G_{\mathbb{C}}x$ and $G_{\mathbb{C}}y$ intersect. This relation is an equivalence relation. Let $M^{ss}//G_{\mathbb{C}}$ be the quotient space and let $\pi_{\mathbb{C}} : M^{ss} \to M^{ss}//G_{\mathbb{C}}$ be the quotient map.

Theorem 2 ([16]). *The inclusion $\phi^{-1}(0) \subset M^{ss}$ induces a homeomorphism $M_0 = \phi^{-1}(0)/G \to M^{ss}//G_{\mathbb{C}}$.*

We say that a subset A of M^{ss} is **saturated** with respect to $\pi_{\mathbb{C}}$ if $\pi_{\mathbb{C}}^{-1}\pi_{\mathbb{C}}(A) = A$.

Proposition 2 ([16]). *At every point of $\phi^{-1}(0)$, there exists a* **holomorphic slice** *D such that the set $G_{\mathbb{C}}D$ is* **saturated** *with respect to the quotient mapping $\pi_{\mathbb{C}}$.*

We identify the spaces $M^{ss}//G_{\mathbb{C}}$ and M_0. We furnish M_0 with a complex-analytic structure such that the quotient map $\pi_{\mathbb{C}}$ is holomorphic. We define a function f defined on an open subset O of M_0 to be holomorphic if the pullback of f to $\pi_{\mathbb{C}}^{-1}(O)$ is holomorphic. Let $\mathcal{O}_{M_0}$ be the sheaf of holomorphic functions on M_0.

Theorem 3 ([16]). *The ringed space $(M_0, \mathcal{O}_{M_0})$ is an analytic space.*

The following theorem describes the property of the stable set $M^s \subset M^{ss}$. If 0 is a regular value of ϕ, then $M^s = M^{ss}$. In general, if $M^s \neq \emptyset$, then M^s is open and dense in M^{ss}.

Theorem 4 ([16]). *If $x \in M$ is stable, then the orbit $G_{\mathbb{C}}x$ is closed in M^{ss}. Let Z be the set of points $m \in \phi^{-1}(0)$ with the property that $d\phi_m$ is surjective. Then the stable set M^s is equal to $F_{\infty}^{-1}(Z)$. Every fiber of $\pi_{\mathbb{C}}|_{M^s}$ consists of a single orbit.*

By this theorem, we see that if a G-orbit $\mathrm{O} = G \cdot x$ in $\phi^{-1}(0)$ has the dimension of G, then only one complex orbit $G_{\mathbb{C}} \cdot x = G_{\mathbb{C}} \cdot \mathrm{O}$ flows to O under the gradient flow of $-\|\phi\|^2$.

The stratification of M_0 as a stratified symplectic space is given by orbit types. Let $p \in M_0$, and let $x \in \pi^{-1}(p)$, where $\pi : \phi^{-1}(0) \to M_0$ is the quotient map. Let (H) be a conjugacy class of closed subgroups of G. Then p is said to be of orbit type (H) if the stabilizer of x is conjugate to H. By [15], the set of all points of orbit type (H) in M_0 is a symplectic manifold.

We can similarly define $G_{\mathbb{C}}$-orbit types. We can show that if $x \in \phi^{-1}(0)$, then the complex stabilizer $(G_{\mathbb{C}})_x$ is equal to the complexification $(G_x)_{\mathbb{C}}$ of the compact stabilizer G_x (see Proposition 1.6 in [16]). By Proposition 1, the fiber $\pi_{\mathbb{C}}^{-1}(p)$ contains a unique closed $G_{\mathbb{C}}$-orbit $G_{\mathbb{C}}x$. Let us say p is of $G_{\mathbb{C}}$-orbit type $(H_{\mathbb{C}})$ if the stabilizer $(G_{\mathbb{C}})_x$ is conjugate to $H_{\mathbb{C}}$ in $G_{\mathbb{C}}$.

Theorem 5 ([16]). *The stratification of M_0 by G-orbit types is identical to the stratification by $G_{\mathbb{C}}$-orbit types. Each stratum $\mathcal{S}$ is a complex manifold and its closure is a complex-analytic subvariety of M_0. The reduced symplectic form on $\mathcal{S}$ is a Kähler form.*

3. Quantization of Kähler manifolds

Let M a connected compact Kähler manifold as in the last section. Assume that the Kähler form ω is integral, i.e., the cohomology class $[\omega/2\pi]$ is an integral cohomology class. Then M is quantizable, i.e., there is a Hermitian line bundle L with compatible connection ∇ such that its curvature is $-\mathrm{i}\omega$. The k-th tensor power $L^{\otimes k}$ of L is a Hermitian line bundle over M with induced Hermitian structure from L. For each k, $L^{\otimes k}$ may be given the structure of a holomorphic line bundle. For each fixed k, the quantum Hilbert space is the space of holomorphic sections of $L^{\otimes k}$ over M, denoted $\mathcal{H}(M, L^{\otimes k})$. Let $\epsilon_\omega = \frac{\omega^n}{n!}$ be the Liouville volume form on M. Then the inner product on $\mathcal{H}(M, L^{\otimes k})$ is usually defined to be

$$\langle s_1, s_2\rangle = (k/2\pi)^{n/2}\int_M (s_1, s_2)\epsilon_\omega,$$

where (s_1, s_2) is the pointwise Hermitian structure on $L^{\otimes k}$.

In this paper, we will study quantizable Kähler manifolds with a holomorphic Hamiltonian Lie group action. The symplectic quotient at value 0 may not be smooth. To adapt to this situation, we will give two definitions of the inner product on $\mathcal{H}(M, L^{\otimes k})$, respectively in Definitions 4 and 5 of Section 13.

4. Quantum reduction

Let M be a connected compact quantizable Kähler manifold. Let G be a connected compact Lie group acting on M holomorphically and in a Hamiltonian fashion with moment map ϕ. The G action lifts to a holomorphic action on the line bundle L preserving the Hermitian structure. Both the G action on M and on L can be analytically continued to holomorphic $G_{\mathbb{C}}$ actions. The G action on L induces G actions on $\mathcal{H}(M, L^{\otimes k})$. Infinitesimally, the action is given by

$$Q_\xi s = \nabla^{(k)}_{X^\xi} s - ik\phi_\xi s, \quad \text{for } \xi \in \mathfrak{g},$$

where $\nabla^{(k)}$ is the induced connection on $L^{\otimes k}$, and ϕ_ξ is the "ξ-moment map component", i.e., $\phi_\xi = \langle\phi, \xi\rangle$. The reduction at quantum level amounts to taking G-invariant holomorphic sections, i.e., taking $\mathcal{H}(M, L^{\otimes k})^G$.

5. Quantization after reduction

Let M be a connected compact quantizable Kähler manifold equipped with a holomorphic Hamiltonian action of a connected compact Lie group G. Let ϕ be the moment map. Let $L_0 = L|_{\phi^{-1}(0)}/G$. Then L_0 is a V-line bundle over M_0, i.e., each point in M_0 has an open neighborhood O which is the quotient of a space $\tilde{O}$ by a finite group Γ such that $L_0|_O$ is the quotient by Γ of a Γ-equivariant line bundle over $\tilde{O}$. As an analytic space, L_0 can be identified with the quotient $L|_{M^{ss}}//G_{\mathbb{C}}$. A holomorphic section of L defined over a $G_{\mathbb{C}}$-invariant open set is G-invariant if and only if it is $G_{\mathbb{C}}$-invariant. Let $\mathcal{L}$ be the sheaf of holomorphic sections of L and define a sheaf $\mathcal{L}'$ on M_0, the sheaf of invariant sections, by letting $\mathcal{L}'(O) = \mathcal{L}(\pi_{\mathbb{C}}^{-1}(O))^{G_{\mathbb{C}}}$ for each open set O of M_0. Then we have

Proposition 3 ([16]). *The sheaf $\mathcal{L}'$ is (the sheaf of sections of) the holomorphic V-line bundle L_0 over $M_0 = M^{ss}//G_{\mathbb{C}}$.*

We take the space of holomorphic sections $\mathcal{H}(M_0, L_0)$ of the V-line bundle L_0 as the quantization of the reduced space M_0.

If we replace L by $L^{\otimes k}$, we have the quantum spaces $\mathcal{H}(M_0, (L^{\otimes k})_0)$.

Since the action of G preserves the Hermitian structure on $L^{\otimes k}$, the Hermitian structure on $L^{\otimes k}$ descends to a Hermitian structure on $(L^{\otimes k})_0$. Let $s \in \mathcal{H}(M, L^{\otimes k})^G$. Then by restricting s to $\phi^{-1}(0)$ and by letting it descend to M_0, we get an element of $\mathcal{H}(M_0, (L^{\otimes k})_0)$. Let us call this linear map A'_k. So, if $x \in \phi^{-1}(0)$, then we have

$$|s|^2(x) = |A'_k s|^2([x]).$$

We still need to define an inner product on $\mathcal{H}(M_0, (L^{\otimes k})_0)$. Denote

$$Z_{(H)} = \{m \in \phi^{-1}(0) : \text{the stabilizer group of } m \text{ is conjugate to } H \subset G\}, \tag{1}$$

and

$$\mathcal{S}_{(H)} = Z_{(H)}/G. \tag{2}$$

As remarked by Sjamaar (see Remark 3.9 in [15]), each $\mathcal{S}_{(H)}$ has finite symplectic volume. For a fixed stratum $\mathcal{S}$ of M_0, let $d_{\mathcal{S}}$ be the complex dimension of $\mathcal{S}$, and let $\epsilon_{\hat{\omega}_{\mathcal{S}}}$ be the volume form on $\mathcal{S}$, where $\hat{\omega}_{\mathcal{S}}$ is the reduced symplectic form on $\mathcal{S}$.

Let $s_1', s_2' \in \mathcal{H}(M_0, (L^{\otimes k})_0)$, and, let (s_1', s_2') be the pointwise Hermitian inner product on $(L^{\otimes k})_0$ inherited from the one on $L^{\otimes k}$. Since there is an open dense connected stratum, say $\mathcal{S}^O$, in M_0, which has full measure, we give the first definition of an inner product on $\mathcal{H}(M_0, (L^{\otimes k})_0)$:

$$\langle s_1', s_2'\rangle_{(1)} = (k/2\pi)^{d_{\mathcal{S}^O}/2} \int_{\mathcal{S}^O} (s_1', s_2')\epsilon_{\hat{\omega}_{\mathcal{S}^O}}. \tag{3}$$

The following second definition of an inner product on $\mathcal{H}(M_0, (L^{\otimes k})_0)$ takes into account all the strata of M_0:

$$\langle s_1', s_2'\rangle_{(2)} = \sum_{\mathcal{S}_{(H)}} (k/2\pi)^{d_{\mathcal{S}_{(H)}}/2} \int_{\mathcal{S}_{(H)}} (s_1', s_2')\epsilon_{\hat{\omega}_{\mathcal{S}_{(H)}}}. \tag{4}$$

If $\mathcal{S}$ is a single point, then the above integral of (s_1', s_2') over $\mathcal{S}$ is just the value of (s_1', s_2') over this point.

6. The linear space isomorphism

In the last section, we defined a linear map A_k' from $\mathcal{H}(M, L^{\otimes k})^G$ to $\mathcal{H}(M_0, (L^{\otimes k})_0)$. We have

Theorem 6 *([16]). Under our hypotheses, the quotient map $\pi_{\mathbb{C}} : M^{ss} \to M_0$ and the inclusion $M^{ss} \subset M$ induce isomorphisms $\mathcal{H}(M_0, (L^{\otimes k})_0) \simeq \mathcal{H}(M^{ss}, L^{\otimes k})^G \simeq \mathcal{H}(M, L^{\otimes k})^G$.*

By Proposition 3, we have the isomorphism $\mathcal{H}(M_0, (L^{\otimes k})_0) \simeq \mathcal{H}(M^{ss}, L^{\otimes k})^G$. The isomorphism $\mathcal{H}(M^{ss}, L^{\otimes k})^G \simeq \mathcal{H}(M, L^{\otimes k})^G$ is based on the observation that the norm of an invariant holomorphic section s of $L^{\otimes k}$ is increasing along the trajectories of $-\mathrm{grad}(\|\phi\|^2)$. It follows that if s is defined on M^{ss}, then $\langle s, s\rangle$ is bounded on M. By Riemann's Extension Theorem, s extends to a G-invariant holomorphic section on M. See [16] for details.

From this theorem, we can deduce that a point $x \in M$ is semistable if there exists an invariant global holomorphic section $s \in \mathcal{H}(M, L^{\otimes l})^G$ for some l such that $s(x) \neq 0$ (see [16]). So the set of unsemistable points is contained in the 0 set of s; therefore it has complex codimension at least one.

7. Half form bundles on *M*

Let $K = \bigwedge^n (T^{1,0}M)^*$ be the canonical bundle of M. A smooth section of K is called an $(n, 0)$-form. We know that the first Chern class of K is $-c_1(M)$. Assume $c_1(M)/2$ is integral. Then the square root $\sqrt{K}$ of the bundle K exists. We fix a choice of $\sqrt{K}$. The group G acts on sections of K. Infinitesimally, a Lie algebra element $\xi \in \mathfrak{g}$ acts on $(n, 0)$-forms by taking the Lie derivative L_{X^ξ} of each form. This induces an action of $\mathfrak{g}$ on half forms by $2(L_{X^\xi}\mu)\mu = L_{X^\xi}(\mu^2)$, where $\mu \in \sqrt{K}$. Since G acts holomorphically on M, we can check that $\mathfrak{g}$ preserves the space of holomorphic sections of K and of $\sqrt{K}$.

Let us define a Hermitian structure on $\Gamma(M, \sqrt{K})$, where $\Gamma(M, \sqrt{K})$ is the space of smooth sections of $\sqrt{K}$. Let $\mu, \nu \in \Gamma(M, \sqrt{K})$ be half forms, then $\mu^2 \wedge \bar{\nu}^2 \in \Gamma(\bigwedge^{2n} T^*(M))$. The volume form ϵ_ω is a global trivializing section of $\bigwedge^{2n} T^*(M)$. So there is a function, denoted (μ, ν), such that

$$\mu^2 \wedge \bar{\nu}^2 = (\mu, \nu)^2 \epsilon_\omega. \tag{5}$$

The function (μ, ν) is defined to be the pointwise inner product of μ and ν.

We use this to define a Hermitian form on $\Gamma(M, L^{\otimes k} \otimes \sqrt{K})$. Let $t_1, t_2 \in \Gamma(M, L^{\otimes k} \otimes \sqrt{K})$ which are locally represented by $t_j(x) = s_j(x)\mu_j(x)$, and we define

$$(t_1, t_2)(x) = (s_1(x), s_2(x))(\mu_1, \mu_2)(x). \tag{6}$$

8. The "push down" of the half form bundle $\sqrt{K}$ to the reduced space M_0

8.1. When the G action on $\phi^{-1}(0)$ is free

Before we come to the general case, let us recall first the procedure given by Hall and Kirwin of pushing down a half form bundle $\sqrt{K}$ of M to a half form bundle $\sqrt{\hat{K}}$ on $M//G$ in the case when the action of G on $\phi^{-1}(0)$ is free.

Let α be a $G_{\mathbb{C}}$-invariant $(n,0)$-form on M. Hall and Kirwin obtained an $(n-d,0)$-form (d is the dimension of G) $\hat{\beta}$ on $M//G$ in the following way. Choose a G-invariant inner product on $\mathfrak{g}$ normalized so that the volume of G with respect to the associated Haar measure is 1. Fix an orthonormal basis $\xi_1, \xi_2, \ldots, \xi_d$ of the Lie algebra $\mathfrak{g}$. Let $X^{\xi_1}, X^{\xi_2}, \ldots, X^{\xi_d}$ be the vector fields they generate on M. For any $x \in M^s$, define

$$\beta = i\left(\bigwedge_j X^{\xi_j}\right)\alpha.$$

One can show that β is basic with respect to the projection map $\pi_{\mathbb{C}}$. So $\beta = \pi_{\mathbb{C}}^*(\hat{\beta})$, where $\hat{\beta}$ is an $(n-d,0)$-form on $M//G$. Let $\mathfrak{B}$ be the map

$$\mathfrak{B}(\alpha) = \hat{\beta}.$$

Conversely, one can construct the inverse map of this push down map. Given an $(n-d,0)$-form $\hat{\beta}$ on $M//G$. The pull back $\beta = \pi_{\mathbb{C}}^*(\hat{\beta})$ is a $G_{\mathbb{C}}$-invariant $(n-d,0)$-form on M^s. One can construct a $G_{\mathbb{C}}$-invariant $(n,0)$-form α on M^s from β. Given a local frame $X^{\xi_1}, X^{\xi_2}, \ldots, X^{\xi_d}, Y_1, \ldots, Y_{n-d}$ for $T_x M^s$, set

$$\alpha(X^{\xi_1}, X^{\xi_2}, \ldots, X^{\xi_d}, Y_1, \ldots, Y_{n-d}) = \pi_{\mathbb{C}}^*\hat{\beta}(Y_1, \ldots, Y_{n-d}),$$

and define α on any other frame by $GL(n,\mathbb{C})$-equivariance and the requirement that α be an $(n,0)$-form. Every other frame is equivalent to a linear combination of frames which are $GL(n,\mathbb{C})$-equivalent to one of the form $W_1, W_2, \ldots, W_d, Y_1, \ldots, Y_{n-d}$ where $W_j = X^{\xi_j}$ or JX^{ξ_j}.

Assume that the $\mathfrak{g}$ action on $\sqrt{K}$ exponentiates to a G action and it is compatible with the G action on K. It can be shown that the G action on $\sqrt{K}$ can be analytically continued to a $G_{\mathbb{C}}$ action. Define a line bundle $\sqrt{\hat{K}}$ over $M//G$ whose fiber is the equivalence class of $\sqrt{K}$ under the $G_{\mathbb{C}}$ action. For a $G_{\mathbb{C}}$-invariant smooth section $\mu \in \Gamma(M, \sqrt{K})^{G_{\mathbb{C}}}$, we define the map

$$B : \Gamma(M, \sqrt{K})^{G_{\mathbb{C}}} \to \Gamma(M//G, \sqrt{\hat{K}})$$

by $(B\mu)^2 = \mathfrak{B}(\mu^2)$.

Since for an $(n,0)$-form α, contracting with $\bigwedge_j X^{\xi_j}$ is the same as contracting with $\bigwedge_j \pi_+ X^{\xi_j}$, where $\pi_+ X^{\xi_j} = \frac{1}{2}(X^{\xi_j} - iJX^{\xi_j})$, and the vector fields $\pi_+ X^{\xi}$ are holomorphic, α is locally holomorphic if and only if $\mathfrak{B}(\alpha)$ is locally holomorphic; and, μ is locally holomorphic if and only if $B(\mu)$ is locally holomorphic.

8.2. When the G action on $\phi^{-1}(0)$ is not necessarily free

Now we come to the general case.

Lemma 1. *Let $\alpha \in \Gamma(M, K)^{G_{\mathbb{C}}}$. Then, α descends to a smooth $(d_{\mathcal{S}}, 0)$-form $\hat{\beta}|_{\mathcal{S}}$ on each smooth stratum $\mathcal{S}$ of M_0 of complex dimension $d_{\mathcal{S}}$. If α is holomorphic, then each $\hat{\beta}|_{\mathcal{S}}$ is holomorphic.*

Proof. Let $Z_{(H)}$ and $\mathcal{S}_{(H)}$ be as in (1) and (2). Take the complex submanifold $G_{\mathbb{C}} \cdot Z_{(H)}$. Let

$$\alpha_| = \alpha|_{G_{\mathbb{C}} \cdot Z_{(H)}}.$$

Then $\alpha_|$ is a $G_{\mathbb{C}}$-invariant $(m,0)$-form on $G_{\mathbb{C}} \cdot Z_{(H)}$, assuming m is the complex dimension of $G_{\mathbb{C}} \cdot Z_{(H)}$.

Case 1. Assume $H = G$. Then $G_{\mathbb{C}} \cdot Z_G = Z_G = Z_G/G = \mathcal{S}_G$. We define $\hat{\beta}|_{\mathcal{S}_G} = \alpha|_{Z_G}$.

Case 2. Assume $H \neq G$. Assume we have chosen a normalized G-invariant inner product on $\mathfrak{g}$. Let $\xi_1, \ldots, \xi_h$ be an orthonormal basis of $\mathfrak{h} = Lie(H)$, expand it to an orthonormal basis of $\mathfrak{g} = Lie(G)$ by joining $\xi_{h+1}, \ldots, \xi_d$. At each point x of $Z_{(H)}$ with stabilizer group H, we define

$$\beta_| = i \left(\bigwedge_{j=h+1,\ldots,d} X^{\xi^j} \right) \alpha_|.$$

We contract the form $\alpha_|$ similarly at the points of $Z_{(H)}$ with stabilizers conjugate to H. So, along $Z_{(H)}$, we have a new form $\beta_|$. Let

$$\beta_{||} = \beta_| |_{Z_{(H)}}.$$

This restriction "cuts off" the $JX^{Ad(G)\cdot\xi^j}$, $j = h+1, \ldots, d$ directions which are normal to $Z_{(H)}$ in $G_{\mathbb{C}} \cdot Z_{(H)}$. Now, $\beta_{||}$ is a smooth G-invariant $(m - d_{G/H})$-form defined on $Z_{(H)}$. By the above contraction and by G-invariance of the form α, clearly, $\beta_{||} = \pi^*(\hat{\beta}_|)$, where $\hat{\beta}_|$ is a $(m - d_{G/H}, 0)$-form on $\mathcal{S}_{(H)}$, and $\pi : Z_{(H)} \to \mathcal{S}_{(H)}$ is the quotient map.

By the above construction, if α is holomorphic, then $\hat{\beta}|_{\mathcal{S}}$ is holomorphic (see the reason we mentioned in Section 8.1). □

Next, we will use the holomorphic slice theorem to see how the forms $\hat{\beta}|_{\mathcal{S}}$'s are related.

Lemma 2. *The forms $\hat{\beta}|_{\mathcal{S}}$'s in Lemma 1 satisfy: if $\mathcal{S} \subset \bar{\mathcal{S}}'$, then $\hat{\beta}|_{\mathcal{S}}$ is obtained from $\hat{\beta}|_{\mathcal{S}'}$ by degenerating some directions.*

Proof. Let $x_0 \in Z_{(H)}$ be a point with stabilizer group H. Take a **saturated** open neighborhood $U = G_{\mathbb{C}} D = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} D$ of x_0 (see Proposition 2), where $H_{\mathbb{C}} = (H)_{\mathbb{C}}$ is the complex stabilizer group of x_0, which is the complexification of H. Split $D = D_1 \times D_2$, where D_1 is the fixed complex subspace of the H action, and therefore the $H_{\mathbb{C}}$ action. So $U = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} (D_1 \times D_2)$. The set

$$U//G_{\mathbb{C}} = (D_1 \times D_2)//H_{\mathbb{C}} = D_1 \times D_2//H_{\mathbb{C}}$$

is a neighborhood of $[x_0]$ in M_0.

The set U is G-equivariantly diffeomorphic to $G \times_H (\sqrt{-1}\mathfrak{m} \times D_1 \times D_2)$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. We pull back (or restrict) the symplectic form ω on M to U. The group $H_{\mathbb{C}}$ acts on D_2 holomorphically. Assume $\phi_|$ is the moment map for the H-action on D_2 with respect to the restricted Kähler form. Then

$$U \cap \phi^{-1}(0) = G \times_H (D_1 \times \phi_|^{-1}(0)).$$

Denote

$$Z'_{(H')} = \{m \in \phi_|^{-1}(0) \subset D_2 : \text{the compact stabilizer group of } m \text{ is conjugate to } H' \subset H\},$$

and recall (1) for $Z_{(H)}$. We have

$$U \cap Z_{(H')} = G \times_H (D_1 \times Z'_{(H')}), \quad \text{in particular, } U \cap Z_{(H)} = G \times_H (D_1 \times 0),$$

where 0 is in the closure of $Z'_{(H')}$. Also

$$G_{\mathbb{C}} \cdot (U \cap Z_{(H')}) = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} (D_1 \times H_{\mathbb{C}} Z'_{(H')}), \quad \text{and,} \quad G_{\mathbb{C}} \cdot (U \cap Z_{(H)}) = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} (D_1 \times 0).$$

The quotients are

$$(U \cap Z_{(H')})/G = D_1 \times Z'_{(H')}/H, \quad \text{which is the same as}$$

$$G_{\mathbb{C}} \cdot (U \cap Z_{(H')})//G_{\mathbb{C}} = D_1 \times (H_{\mathbb{C}} Z'_{(H')})//H_{\mathbb{C}},$$

and

$$(U \cap Z_{(H)})/G = D_1 \times 0, \quad \text{which is the same as } G_{\mathbb{C}} \cdot (U \cap Z_{(H)})//G_{\mathbb{C}}.$$

Now, restricting to the open set U, resp., $U//G_{\mathbb{C}}$, the relation between $Z_{(H)}$ and $Z_{(H')}$, resp., the relation between $\mathcal{S}_{(H)}$ and $\mathcal{S}_{(H')}$, is clear. In the open set U, doing the specified restricting, contracting, restricting again, and pushing down of the form $\alpha_|$ as in the proof of Lemma 1, we see that if we use local coordinates, and if $\hat{\beta}|_{\mathcal{S}_{(H')}} = g(w_1, \ldots, w_k) dw_1 \wedge \cdots \wedge dw_k$, then $\hat{\beta}|_{\mathcal{S}_{(H)}} = g(w_{i_1}, \ldots, w_{i_j}, 0, \ldots, 0) dw_{i_1} \wedge \cdots \wedge dw_{i_j}$, where $\{i_1, \ldots, i_j\} \subset \{1, \ldots, k\}$. □

Let us simply use $\hat{\beta}$ to denote this family of forms on M_0 we have obtained. It has different dimensions on different dimensional strata.

Let us denote the above push down map by

$$\mathfrak{B}' : \mathfrak{B}'(\alpha) = \hat{\beta}.$$

Remark 1. Let us use local coordinates on $U = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} (D_1 \times D_2)$ based at a point x with stabilizer group H to see the push down map described in the proof of Lemma 1. Let z_0 be the coordinate along the $G_{\mathbb{C}}$-orbit direction, and let $(z_1, z_2) \in D_1 \times D_2$ be the coordinate in the transversal direction. Then, for instance, we may write a $G_{\mathbb{C}}$-invariant $(n, 0)$-form $\alpha = f(z_0, z_1, z_2) dz_0 \wedge dz_1 \wedge dz_2$ locally, where f is a $G_{\mathbb{C}}$-invariant function. Restricting α to $G_{\mathbb{C}} \cdot (U \cap Z_{(H)})$, we get $\alpha_| = f(z_0, z_1, 0) dz_0 \wedge dz_1$. The contraction gives $\beta_| = f(z_0, z_1, 0) dz_1$ (up to a sign), and the restriction of $\beta_|$ to $U \cap Z_{(H)}$ gives $\beta_{||} = f(x_0, z_1, 0) dz_1$. By the G-invariance of α, $\beta_{||} = \pi^*(\hat{\beta}_|)$, where $\hat{\beta}_| = f([x], z_1, 0) dz_1$ is a local form on D_1 which is a neighborhood of $[x]$ in $\mathcal{S}_{(H)}$. Notice that, conversely, if we have such a $(d_{\mathcal{S}}, 0)$-form $\hat{\beta}_|$ on $\mathcal{S}_{(H)}$, we can lift it to a $(d_{\mathcal{S}} + d_{G/H}, 0)$-form on $G_{\mathbb{C}} \cdot Z_{(H)}$ by using $G_{\mathbb{C}}$-invariance and by “growing back” the coordinate z_0.

Remark 2. If G acts freely on $\phi^{-1}(0)$, then $\phi^{-1}(0) = Z_1$, where $1 \in G$ is the identity element. By the holomorphic slice theorem, a saturated neighborhood of each point $x \in \phi^{-1}(0)$ is biholomorphic to $U = G_{\mathbb{C}} \times D$. So $U \cap \phi^{-1}(0) = G \times D$, and $D = (U \cap \phi^{-1}(0))/G = U//G_{\mathbb{C}}$ is biholomorphic to a neighborhood of $[x]$ in M_0. In our point of view, we first restrict a $G_{\mathbb{C}}$-invariant $(n, 0)$-form α to U; then we contract the form at the points in $U \cap \phi^{-1}(0)$ with the generating vector fields of the free G-action; then we restrict the resulting form to $U \cap \phi^{-1}(0)$ and push it down to D by the quotient map $\pi : U \cap \phi^{-1}(0) \to D$. One may see this in local coordinates as we did in the last remark. In the point of view of Hall and Kirwin, they contract the form α at the points in U with the generating vector fields of the free G-action (G acts freely on U), and then push down the resulting form to D by the quotient map $\pi_{\mathbb{C}} : U \to U//G_{\mathbb{C}} = D$. We see that the results are the same. Their pulling back of an $(n', 0)$-form (let $n' = \dim(D)$) on D to a $G_{\mathbb{C}}$-invariant $(n, 0)$-form to U is just done by using the $G_{\mathbb{C}}$-action and by “adding” the $G_{\mathbb{C}}$-orbit direction.

We define $\hat{K}$ to be $K//G_{\mathbb{C}} = K|_{\phi^{-1}(0)}/G$, and we define $\sqrt{\hat{K}}$ to be $\sqrt{K}//G_{\mathbb{C}} = \sqrt{K}|_{\phi^{-1}(0)}/G$. Sections of $\hat{K}$ over $M//G$ are “stratified forms” $\hat{\beta}$ whose restriction to each stratum $\mathcal{S}$ of complex dimension $d_{\mathcal{S}}$ is a smooth $(d_{\mathcal{S}}, 0)$-form. If O is a small open set in $M//G$, a section of $\hat{K}$ over O looks like $\hat{f}_| dw_1 \wedge dw_2 \wedge \cdots \wedge dw_r$ on the open dense stratum, and looks like $\hat{f}_| dw_{i_1} \wedge \cdots \wedge dw_{i_j}$ for some subset $\{i_1, \ldots, i_j\}$ of $\{1, \ldots, r\}$ on other strata, where $\pi_{\mathbb{C}}^*(\hat{f})$ is a $G_{\mathbb{C}}$-invariant function on $\pi_{\mathbb{C}}^*(O)$.

We defined the map $\mathfrak{B}' : \Gamma(M, K)^{G_{\mathbb{C}}} \to \Gamma(M//G, \hat{K})$. Using this map, we define a linear map

$$B' : \Gamma(M, \sqrt{K})^{G_{\mathbb{C}}} \to \Gamma(M//G, \sqrt{\hat{K}})$$

such that

$$(B'\mu)^2 = \mathfrak{B}'(\mu^2). \tag{7}$$

Using the map A'_k and the map B', for each k, we define a linear map

$$B'_k : \Gamma(M, L^{\otimes k} \otimes \sqrt{K})^G \to \Gamma(M//G, (L^{\otimes k})_0 \otimes \sqrt{\hat{K}}),$$

unique up to an overall sign, such that

$$B'_k(s \otimes \mu) = A'_k(s) \otimes B'(\mu),$$

where $s \in \Gamma(L^{\otimes k})^G$ and $\mu \in \Gamma(\sqrt{K})^G$.

Next, we give an argument for the facts

$$\mathcal{H}(M//G, \hat{K}) = \mathfrak{B}'(\mathcal{H}(M^{ss}, K)^{G_{\mathbb{C}}}),$$

and

$$\mathcal{H}(M//G, \sqrt{\hat{K}}) = B'(\mathcal{H}(M^{ss}, \sqrt{K})^{G_{\mathbb{C}}}).$$

A holomorphic section of K (or of $\sqrt{K}$) defined over a $G_{\mathbb{C}}$-invariant open set is G-invariant if and only if it is $G_{\mathbb{C}}$-invariant. Let $\mathcal{K}$ (or $\sqrt{\mathcal{K}}$) be the sheaf of holomorphic sections of K (or of $\sqrt{K}$); and define a sheaf $\mathcal{K}'$ (or $\sqrt{\mathcal{K}'}$) on M_0, by letting $\mathcal{K}'(O) = \mathfrak{B}'(\mathcal{K}(\pi_{\mathbb{C}}^{-1}(O))^{G_{\mathbb{C}}})$ (or by letting $\sqrt{\mathcal{K}'}(O) = B'(\sqrt{\mathcal{K}}(\pi_{\mathbb{C}}^{-1}(O))^{G_{\mathbb{C}}})$) for each open set O of M_0. Using our results above and combining the argument of the proof of Proposition 3, we have the following:

Proposition 4. *The sheaf $\mathcal{K}'$ (or $\sqrt{\mathcal{K}'}$) is the sheaf of holomorphic sections of the stratified-line bundle $\hat{K}$ (or $\sqrt{\hat{K}}$) over $M_0 = M^{ss}//G_{\mathbb{C}}$.*

So we have proved the following

Theorem 7. *Let $\sqrt{\hat{K}} = \sqrt{K}//G_{\mathbb{C}}$. There exists a linear map $B' : \Gamma(M, \sqrt{K})^{G_{\mathbb{C}}} \to \Gamma(M//G, \sqrt{\hat{K}})$, unique up to an overall sign, such that for each $\mu \in \Gamma(M, \sqrt{K})^{G_{\mathbb{C}}}$, $B'(\mu)$ is a "stratified form" on $M//G$ such that for each stratum $\mathcal{S}$ of $M//G$ with complex dimension $d_{\mathcal{S}}$, $(B'(\mu))^2|_{\mathcal{S}}$ is a $(d_{\mathcal{S}}, 0)$- form on $\mathcal{S}$, and these forms are related by suitable degenerating of directions from higher dimensional strata to lower dimensional strata. Moreover, $\mathcal{H}(M//G, \sqrt{\hat{K}}) = B'(\mathcal{H}(M^{ss}, \sqrt{K})^{G_{\mathbb{C}}})$.*

Consequently, for each k, there exists a linear map $B'_k : \Gamma(M, L^{\otimes k} \otimes \sqrt{K})^{G_{\mathbb{C}}} \to \Gamma(M//G, (L^{\otimes k})_0 \otimes \sqrt{\hat{K}})$, unique up to an overall sign, such that

$$B'_k(s \otimes \mu) = A'_k(s) \otimes B'(\mu)$$

for $s \in \Gamma(M, L^{\otimes k})^{G_{\mathbb{C}}}$ and $\mu \in \Gamma(M, \sqrt{K})^{G_{\mathbb{C}}}$, and such that $\mathcal{H}(M//G, (L^{\otimes k})_0 \otimes \sqrt{\hat{K}}) = B'_k(\mathcal{H}(M^{ss}, L^{\otimes k} \otimes \sqrt{K})^{G_{\mathbb{C}}})$.

We end this section by giving the definition of a pointwise Hermitian structure on $\Gamma(M//G, \sqrt{\hat{K}})$. Let $\mu', \nu' \in \Gamma(M//G, \sqrt{\hat{K}})$. We define a Hermitian structure on $\Gamma(M//G, \sqrt{\hat{K}})$ stratum-wise as

$$(\mu')^2 \wedge (\bar{\nu'})^2|_{\mathcal{S}} = (\mu', \nu')^2|_{\mathcal{S}}\, \epsilon_{\hat{\omega}_{\mathcal{S}}}, \tag{8}$$

where $\epsilon_{\hat{\omega}_{\mathcal{S}}}$ is the volume form on the stratum $\mathcal{S}$ of $M//G$.

9. Modified linear space isomorphism

The following theorem gives the growth of the pointwise norm square of a G-invariant holomorphic section and a modified G-invariant holomorphic section along the gradient curves of the moment map components. We need this theorem to prove Theorem 9, and we will need this theorem in the subsequent sections.

Theorem 8. *Let $s \in \mathcal{H}(M, L^{\otimes k})^G$ and let $r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. Let $y_0 \in M$, and let H be its stabilizer group. Let $\mathfrak{h} = Lie(H)$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g} = Lie(G)$ (assuming we have chosen a G-invariant metric on $\mathfrak{g}$). Then for $0 \neq \xi \in \mathfrak{m}$, we have*

(a) $|s|^2(\mathrm{e}^{\mathrm{i}\xi} \cdot y_0) = |s|^2(y_0) \exp\left\{-\int_0^1 2k\phi_\xi(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0)\mathrm{d}t\right\}$,

(b) $|r|^2(\mathrm{e}^{\mathrm{i}\xi} \cdot y_0) = |r|^2(y_0) \exp\left\{-\int_0^1 (2k\phi_\xi(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0) + \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega}(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0))\mathrm{d}t\right\}$.

If we let $f(\xi, y_0) := 2\int_0^1 \phi_\xi(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0)\mathrm{d}t$, then as a function of $\xi \in \mathfrak{m}$, $f(\xi, y_0)$ achieves its unique minimum at $\xi = 0$. The Hessian of $f(\xi, y_0)$ at $\xi = 0$ is given by

$D_{\xi_1} D_{\xi_2} f(\xi, y_0)|_{\xi=0} = 2B_{y_0}(JX^{\xi_1}, JX^{\xi_2})$, $\xi_1, \xi_2 \in \mathfrak{m}$.

Proof. See the proof of Theorem 4.1 in [9]. Modify the proof by noticing the following: for $y_0 \in M$, if H is the stabilizer group of y_0, and if $0 \neq \xi \in \mathfrak{h} = Lie(H)$, then $X^\xi(y_0) = 0$, so $JX^\xi(y_0) = 0$ as well; therefore $\mathrm{e}^{\mathrm{i}\xi} \cdot y_0 = y_0$. □

Using (b) of the above theorem, we obtain the following modified linear space isomorphism:

Theorem 9. *For k sufficiently large, the map*

$$B'_k : \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G \to \mathcal{H}(M//G, (L^{\otimes k})_0 \otimes \sqrt{\hat{K}})$$

is bijective.

Proof. We use a similar argument as used by Guillemin and Sternberg in [6], by Sjamaar in [16], and by Hall and Kirwin in [9] (the proof of Theorem 3.2 in [9]).

By Theorems 6 and 7, elements in $\mathcal{H}(M//G, (L^{\otimes k})_0 \otimes \sqrt{\hat{K}})$ lift to elements in $\mathcal{H}(M^{ss}, L^{\otimes k} \otimes \sqrt{K})^G$.

The map is injective because the two holomorphic sections which agree on the semistable set, which is open and dense in M, must be equal.

Let $\hat{r} \in \mathcal{H}(M//G, (L^{\otimes k})_0 \otimes \sqrt{\hat{K}})$, and let $r \in \mathcal{H}(M^{ss}, L^{\otimes k} \otimes \sqrt{K})^G$ be its lift. We only need to show that $|r|^2$ remains bounded as we approach the unsemistable set (which is of complex codimension at least one); the Riemann Extension Theorem will imply that r extends holomorphically to all of M.

By Theorem 8(b), for $y_0 \in M^{ss}$ with stabilizer group H, and for $\xi \in \mathfrak{m}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|r|^2(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0) = |r|^2(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0)(-2k\phi_\xi(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0) - \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega}(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0)).$$

Notice that, for $\xi \in \mathfrak{h} = Lie(H)$, $|r|^2(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0) = |r|^2(y_0)$, and so $\frac{\mathrm{d}}{\mathrm{d}t}|r|^2(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0) = 0$.

By the compactness of M and by the compactness of the set $\{\xi \in \mathfrak{g} : |\xi| = 1\}$, $\frac{L_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega}$ is bounded uniformly for all $\xi \in \mathfrak{g}$ with $|\xi| = 1$ and at all points in M.

By the monotonicity of $\phi_\xi(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0)$ in t for $\xi \in \mathfrak{m}$, and by the above fact about $\xi \in \mathfrak{h}$, we see that for all sufficiently large k, $\frac{\mathrm{d}}{\mathrm{d}t}|r|^2(\mathrm{e}^{\mathrm{i}t\xi} \cdot y_0) \leq 0$ for all $y_0 \in M^{ss}$, all $\xi \in \mathfrak{g}$ with $|\xi| = 1$, and all $t \geq 1$. It follows that the r obtained extends holomorphically to all of M. □

10. The pointwise norms of the modified sections

Theorem 10. *Suppose $r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. Let $x_0 \in \phi^{-1}(0)$ be a point with stabilizer group H. So $[x_0] \in \mathcal{S}_{(H)} = Z_{(H)}/G$. Then, if $H = G$,*

$$\pi^*(|B'_k r|^2([x_0])) = |r|^2(x_0);$$

otherwise

$$\pi^*(|B'_k r|^2([x_0])) = 2^{-d_{G/H}/2}\mathrm{vol}(G \cdot x_0)|r|^2(x_0),$$

where $d_{G/H}$ is the dimension of G/H.

By modifying the proofs of Lemmas 3.4 and 3.5 in [9], we can prove the following two lemmas.

Lemma 3. *Let $x \in M$ be a point with isotropy group H. Let $G \cdot x$ be the orbit through x. Assume that we have chosen a normalized G-invariant inner product on $\mathfrak{g}$. Let $\xi_1, \ldots, \xi_h$ be an orthonormal basis of $\mathfrak{h} = Lie(H)$. If $\mathfrak{h} \neq \mathfrak{g}$, we expand $\xi_1, \ldots, \xi_h$ to an orthonormal basis of $\mathfrak{g}$ by joining $\xi_{h+1}, \ldots, \xi_d$. Then, the function $\sqrt{\det_{j,k=h+1,\ldots,d}(B(X^{Ad(g)\xi_j}, X^{Ad(g)\xi_k}))_{g\cdot x}}$ is a constant along the G orbit $G \cdot x$, and*

$$\mathrm{vol}(G \cdot x) = \sqrt{\det_{j,k=h+1,\ldots,d}(B(X^{\xi_j}, X^{\xi_k}))_x}.$$

Lemma 4. *Let $x_0 \in \phi^{-1}(0)$. Assume x_0 has isotropy group $H \neq G$. Choose a basis as in the Lemma above. Let $Z^j = \pi_+ X^{\xi_j} = \frac{1}{2}(X^{\xi_j} - iJX^{\xi_j})$, for $j = h+1, \ldots, d$. Let $\mathcal{S}_{(H)} = Z_{(H)}/G$, and let $d_{\mathcal{S}_{(H)}} = \dim_{\mathbb{C}}(\mathcal{S}_{(H)})$. Then $\dim_{\mathbb{C}}(G_{\mathbb{C}} \cdot Z_{(H)}) = d_{\mathcal{S}_{(H)}} + d_{G/H}$. Let $\omega_| = \omega|_{G_{\mathbb{C}} \cdot Z_{(H)}}$, and let $(\epsilon_{\omega_|})_| = \frac{\omega_|^{(d_{\mathcal{S}_{(H)}}+d_{G/H})}}{(d_{\mathcal{S}_{(H)}}+d_{G/H})!}$. Then*

$$i\left(\bigwedge_j Z^j\right) \circ i\left(\bigwedge_k \bar{Z}^k\right)(\epsilon_{\omega_|})_|(x_0)\Bigg|_{Z_{(H)}} = 2^{-d_{G/H}}\mathrm{vol}(G \cdot x_0)^2\, \frac{\omega^{d_{\mathcal{S}_{(H)}}}}{d_{\mathcal{S}_{(H)}}!}(x_0)\Bigg|_{Z_{(H)}}.$$

The proof of Lemma 4 uses the result of Lemma 3. Now, we use Lemma 4 to prove Theorem 10.

Proof. Near x_0, we can write $r = s\mu$, where s is a local G-invariant holomorphic section of $L^{\otimes k}$ and μ is a local G-invariant holomorphic section of $\sqrt{K}$. Let $\alpha = \mu^2$, and let $\alpha_| = \alpha|_{G_{\mathbb{C}} \cdot Z_{(H)}}$. Then, by (8) and by (7), we have

$$(*) \quad \pi^*((B'\mu, B'\mu)^2 \epsilon_{\hat{\omega}_{\mathcal{S}_{(H)}}}([x_0])) = \pi^*(\mathfrak{B}'(\alpha)([x_0]) \wedge \mathfrak{B}'(\bar{\alpha})([x_0])|_{\mathcal{S}_{(H)}}).$$

Case 1. Assume $H = G$. Then, by the construction of the map $\mathfrak{B}'$ (see the proof of Lemma 1),

$$(*) = (\alpha_| \wedge \bar{\alpha}_|)(x_0)|_{Z_G} \overset{\text{by (5)}}{=} (\mu,\mu)^2 (\epsilon_{\omega_|})_|(x_0)|_{Z_G} = (\mu,\mu)^2\, \frac{\omega^{d_{\mathcal{S}_G}}}{d_{\mathcal{S}_G}!}(x_0)\Bigg|_{Z_G}.$$

Case 2. Assume $H \neq G$. Then, by construction of the map $\mathfrak{B}'$,

$$(*) = \left(i\left(\bigwedge_{j=h+1,\ldots,d} Z^j\right)\alpha_|(x_0) \wedge i\left(\bigwedge_{k=h+1,\ldots,d} \bar{Z}^k\right)\bar{\alpha}_|(x_0)\right)\Bigg|_{Z_{(H)}}.$$

Since α is holomorphic, $i(\pi_+ X^\xi)\bar{\alpha} = i(\pi_- X^\xi)\alpha = 0$. So the above

$$= \left(i\left(\bigwedge_{j=h+1,\ldots,d} Z^j \wedge \bigwedge_{k=h+1,\ldots,d} \bar{Z}^k\right)(\alpha_| \wedge \bar{\alpha}_|)(x_0)\right)\Bigg|_{Z_{(H)}}$$

$$\overset{\text{by (5)}}{=} i\left(\bigwedge_{j=h+1,\ldots,d} Z^j \wedge \bigwedge_{k=h+1,\ldots,d} \bar{Z}^k\right)((\mu,\mu)^2(\epsilon_{\omega_|})_|)(x_0)\Bigg|_{Z_{(H)}}$$

$$= (\mu,\mu)^2(x_0) 2^{-d_{G/H}} \mathrm{vol}(G \cdot x_0)^2\, \frac{\omega^{d_{\mathcal{S}_{(H)}}}}{d_{\mathcal{S}_{(H)}}!}(x_0)\Bigg|_{Z_{(H)}} \quad \text{by Lemma 4.}$$

In the above, we used the fact that, if we do $\alpha \wedge \bar{\alpha} = (\mu,\mu)^2 \epsilon_\omega$ on M, we get a function $(\mu,\mu)^2$ on M; the value $(\mu,\mu)^2(x_0)$ is the same as the value $(\mu,\mu)_|^2(x_0)$ of the function $(\mu,\mu)_|^2$ obtained by doing $\alpha_| \wedge \bar{\alpha}_| = (\mu,\mu)_|^2 (\epsilon_{\omega_|})_|$ on the Kähler submanifold $G_{\mathbb{C}} \cdot Z_{(H)}$.

In both cases, dividing by $\pi^* \epsilon_{\hat{\omega}_{\mathcal{S}_{(H)}}} = \omega^{d_{\mathcal{S}_{(H)}}}/(d_{\mathcal{S}_{(H)}})!|_{Z_{(H)}}$, taking the square root and using the fact that $|A'_k s|^2([x_0]) = |s|^2(x_0)$, we obtain the result. □

11. Norms of sections in the quantum spaces

11.1. The coarea formula

The coarea formula was cited in [9]. For convenience, we also include this formula here (see [4], pg. 159–160).

Lemma 5. *Let Q and R be smooth Riemannian manifolds with $\dim(Q) \geq \dim(R)$, and let $p : Q \to R$. Then for any $f \in L^1(Q)$, one has*

$$\int_Q \mathcal{J}_p f \,\mathrm{dvol}(Q) = \int_R \mathrm{dvol}(R)(y) \int_{p^{-1}(y)} (f|_{p^{-1}(y)}) \mathrm{dvol}(p^{-1}(y)),$$

where the Jacobian is $\mathcal{J}_p := \sqrt{\det(p_* \circ p_*^{adj})}$.

For instance, consider $G_{\mathbb{C}} \cdot Z_{(H)}$ or $G_{\mathbb{C}} \cdot S_i$ occuring in Lemma 8 or in Lemma 9 of Section 11.2. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h} = Lie(H)$ or of $\mathfrak{h}'$ in $\mathfrak{g}$. Denote $Z_{(H)}$ or S_i simply as S. Let $\Lambda : \mathfrak{m} \times S \to G_{\mathbb{C}} \cdot S$ be the diffeomorphism $\Lambda(\xi, u) = e^{iAd(g)\xi} \cdot u$, if u has infinitesimal isotropy Lie algebra $Ad(g)\mathfrak{h}$ or $Ad(g)\mathfrak{h}'$. The volume element of $G_{\mathbb{C}} \cdot S$ inherited from M at a point $(\xi,\ g \cdot u)$, where u has isotropy Lie algebra $\mathfrak{h}$ or $\mathfrak{h}'$, decomposes as

$$\Lambda^*(\mathrm{dvol}(G_{\mathbb{C}} \cdot S))_{(\xi,\ g\cdot u)} = \tau(\xi, u)\mathrm{dvol}(\mathfrak{m}) \wedge \mathrm{dvol}(S)_{g\cdot u} \tag{9}$$

for some G-invariant smooth Jacobian function τ, where $\mathrm{dvol}(\mathfrak{m})$ is the Lebesgue measure on $\mathfrak{m}$.

11.2. Norms of the sections in the quantum spaces

In this section, we compute the norms of the sections in the quantum spaces. The main result of this section is Theorem 11.

Let $Z_{(H)}$ be as in (1). Then

$$M^{ss} = \bigcup_{(H)} F_\infty^{-1}(Z_{(H)}),$$

where F_∞ is the limit map of the flow F_t of the gradient of $-\|\phi\|^2$. By dividing further into connected components for each $Z_{(H)}$, we assume that each $Z_{(H)}$ is connected. Since there is an open dense connected stratum $Z_{(H)}$ (for some H) in $\phi^{-1}(0)$ ([15]), there is an open dense connected set $F_\infty^{-1}(Z_{(H)})$ in M^{ss}. We will compute the integral of the pointwise norm square of the sections over each $F_\infty^{-1}(Z_{(H)})$ with H varyng. This integral will relate to the integral over $\mathcal{S}_{(H)}$ (see (2) for the definition of $\mathcal{S}_{(H)}$) of the pointwise norm square of the descended section. In particular, if $Z_{(H)}$ or $\mathcal{S}_{(H)}$ is a single point, then the integral on it is regarded as the pointwise norm square of the sections over this point.

Now, let us take a look at $F_\infty^{-1}(Z_{(H)})$. By 2 and 3 of Proposition 1, $G_{\mathbb{C}} \cdot Z_{(H)} \subseteqq F_\infty^{-1}(Z_{(H)})$. By the holomorphic slice theorem or by Theorem 4, we have

Lemma 6. *If $d\phi_x$ is surjective for all $x \in Z_{(H)}$ (H is necessarily finite), then $F_\infty^{-1}(Z_{(H)}) = G_{\mathbb{C}} \cdot Z_{(H)}$.*

Now, we assume that $F_\infty^{-1}(Z_{(H)})$ is strictly larger than $G_{\mathbb{C}} \cdot Z_{(H)}$. Then $F_\infty^{-1}(Z_{(H)})$ contains complex orbits whose closures contain the complex orbits in $G_{\mathbb{C}} \cdot Z_{(H)}$. We decompose the set $F_\infty^{-1}(Z_{(H)}) - G_{\mathbb{C}} \cdot Z_{(H)}$ into a disjoint union of connected $G_{\mathbb{C}}$-invariant complex submanifolds, each of which has the same infinitesimal compact orbit type, say $(\mathfrak{h}')$. Let $M^{(H)}_{(\mathfrak{h}')}$ denote one of these invariant complex submanifolds.

Lemma 7. *Assume that $F_\infty^{-1}(Z_{(H)})$ is strictly larger than $G_{\mathbb{C}} \cdot Z_{(H)}$. We decompose $F_\infty^{-1}(Z_{(H)}) - G_{\mathbb{C}} \cdot Z_{(H)} = \bigcup_{\mathfrak{h}'} M^{(H)}_{(\mathfrak{h}')}$, where $\bigcup_{\mathfrak{h}'} M^{(H)}_{(\mathfrak{h}')}$ is a disjoint union with each $M^{(H)}_{(\mathfrak{h}')}$ being a connected $G_{\mathbb{C}}$-invariant complex submanifold of a certain infinitesimal compact orbit type $(\mathfrak{h}')$. Then $0 \notin \phi(M^{(H)}_{(\mathfrak{h}')})$ and $\dim(\mathfrak{h}') < \dim(H)$. Therefore, for any possible H, if $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{g}$, then $\dim(\mathfrak{m}) > 0$.*

Proof. By 2 of Proposition 1, the complex orbits in $M^{(H)}_{(\mathfrak{h}')}$ do not intersect with $\phi^{-1}(0)$. So $0 \notin \phi(M^{(H)}_{(\mathfrak{h}')})$.

By Theorem 1, a neighborhood U of $x \in Z_{(H)}$ in M is G-equivariantly biholomorphic to $G_{\mathbb{C}} \times_{H_{\mathbb{C}}} D$. Split $D = D_1 \times D_2$, where D_1 is the complex subspace fixed by H and $H_{\mathbb{C}}$. Let $\phi_|$ be the moment map of the H action on D_2 with respect to the restricted Kähler form. Then $U \cap \phi^{-1}(0) = G \times_H (D_1 \times \phi_|^{-1}(0))$, $U \cap Z_{(H)} = G \times_H (D_1 \times 0)$, and $U \cap G_{\mathbb{C}} \cdot Z_{(H)} = G_{\mathbb{C}} \times_{H_{\mathbb{C}}} (D_1 \times 0)$. By the assumption, $D_2 \neq \emptyset$ and $M^{(H)}_{(\mathfrak{h}')} \cap D_2 \neq \emptyset$. The set $M^{(H)}_{(\mathfrak{h}')} \cap D_2$ is an H-invariant subset of D_2 consisting of points with isotropy Lie algebra $\mathfrak{h}' \subset \mathfrak{h} = Lie(H)$ (A group H' such that $Lie(H') = \mathfrak{h}'$ is a subgroup of H, since any point in D_2 has its isotropy group being a subgroup of H).

Since $\phi(M^{(H)}_{(\mathfrak{h}')})$ does not intersect 0, $\phi_|(M^{(H)}_{(\mathfrak{h}')} \cap D_2)$ does not intersect 0. One only needs to argue when H is not connected and when $\dim(H') = \dim(H)$, and exclude this possibility by using the fact that a finite group action does not contribute to the moment map $\phi_|$. □

By the definition of the moment map, for $x \in M$ with isotropy Lie algebra $\mathfrak{h}'$, the image of $d\phi_x : T_x M \to \mathfrak{g}^*$ is the annihilator in $\mathfrak{g}^*$ of $\mathfrak{h}'$. So the image $\phi(M^{(H)}_{(\mathfrak{h}')})$ intersects with a closed positive Weyl chamber at a certain dimension.

This image may lie on one or more than one open faces of the moment polytope $\triangle$ of ϕ. These faces form a connected set since we took $M^{(H)}_{(\mathfrak{h}')}$ to be connected. For a non-Abelian Lie group action, the moment polytope is defined to be the intersection of the image of the moment map with a fixed closed positive Weyl chamber. The faces of the moment polytope are caused by symplectic submanifolds with different isotropy groups. One should distinguish the faces of the moment polytope with the faces of the Weyl chamber.

Lemma 8. *Assume that $\phi(M^{(H)}_{(\mathfrak{h}')})$ only lies on one open face $\mathcal{F}_0$ of $\triangle$. Then* $\dim(\mathcal{F}_0) > 0$. *Let $0 \neq a_0 \in \phi(M^{(H)}_{(\mathfrak{h}')}) \subset \mathcal{F}_0$ be a value. Then, we can write $M^{(H)}_{(\mathfrak{h}')} = G_{\mathbb{C}} \cdot S_0$, where $S_0 \subseteqq S_{(\mathfrak{h}')} = \{x \in \phi^{-1}(G \cdot a_0) : x$ has isotropy Lie algebra type $(\mathfrak{h}')\}$ and S_0 is G-invariant.*

Proof. Since $F_\infty(M^{(H)}_{(\mathfrak{h}')}) \subset \phi^{-1}(0)$, there are points in $\phi(M^{(H)}_{(\mathfrak{h}')})$ arbitrarily near 0. Since $\phi(M^{(H)}_{(\mathfrak{h}')})$ is connected, dim $(\mathcal{F}_0) > 0$.

Since $S_0 \subset M^{(H)}_{(\mathfrak{h}')}$, and since $M^{(H)}_{(\mathfrak{h}')}$ is $G_{\mathbb{C}}$-invariant, we have $G_{\mathbb{C}} \cdot S_0 \subset M^{(H)}_{(\mathfrak{h}')}$. Conversely, if $x \in M^{(H)}_{(\mathfrak{h}')}$, then $\phi(x) \in G \cdot \mathcal{F}_0$. Without loss of generality, we assume the isotropy Lie algebra of x is $\mathfrak{h}'$ and $\phi(x) = b \in \mathcal{F}_0$. If $b = a_0$, then $x \in S_{(\mathfrak{h}')}$. If $b \neq a_0$, then x can be reached by the flow line of JX^ξ from a point in $\phi^{-1}(a_0)$, where ξ is a vector in $\mathcal{F}_0$ pointing from a_0 to b. So $x \in G_{\mathbb{C}} \cdot S_{(\mathfrak{h}')}$. □

Corollary 1. *Let a be any point on the face $\mathcal{F}_0$, and let $S \subseteqq \{x \in \phi^{-1}(G \cdot a) : x$ has isotropy Lie algebra type $(\mathfrak{h}')\}$. Then $G_{\mathbb{C}} \cdot S_0 = G_{\mathbb{C}} \cdot S = M^{(H)}_{(\mathfrak{h}')}$.*

Proof. We have $\mathcal{F}_0 \subset \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{g}$ which is identified with the annihilator of $\mathfrak{h}'$ in $\mathfrak{g}^*$. The image $\phi(G_{\mathbb{C}} \cdot S_0)$ must cover the face $\mathcal{F}_0$. So $a \in \phi(M^{(H)}_{(\mathfrak{h}')})$. □

Lemma 9. *Assume $\phi(M^{(H)}_{(\mathfrak{h}')})$ lies on more than one faces of $\triangle$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_p$ be the ones whose closures contain 0. Let $0 \neq a_i \in \mathcal{F}_i$, $i = 1, \ldots, p$, and let*

$$S_i \subseteqq \{x \in \phi^{-1}(G \cdot a_i) : x \text{ has isotropy Lie algebra type}(\mathfrak{h}')\}.$$

If $\mathcal{F}_k$ is in the closure of $\mathcal{F}_i$, then $G_{\mathbb{C}} \cdot S_k \subset G_{\mathbb{C}} \cdot S_i$. Moreover, we can write $M^{(H)}_{(\mathfrak{h}')} = \cup_{i\in I}(G_{\mathbb{C}} \cdot S_i)$, where I is the subset of $\{1, \ldots, p\}$ such that $\mathcal{F}_{i\in I}$ are the top dimensional faces among the $\mathcal{F}_i$'s.

Proof. We have $\mathcal{F}_k \subset \bar{\mathcal{F}}_i \subset \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{g}$ which is identified with the annihilator of $\mathfrak{h}'$ in $\mathfrak{g}^*$. Since the points in S_k have isotropy Lie algebra $(\mathfrak{h}')$, the $G_{\mathbb{C}}$ action (or the $i(\mathfrak{m})$ action) will take the points in S_k out and merge them into $G_{\mathbb{C}} \cdot S_i$. Or, equivalently, the moment map value increases along the flow lines of JX^ξ, where $\xi \in \mathfrak{m}$ is orthogonal to $\mathcal{F}_k$. This proves $G_{\mathbb{C}} \cdot S_k \subset G_{\mathbb{C}} \cdot S_i$.

So, using Lemma 8, $\cup_{i\in I}(G_{\mathbb{C}} \cdot S_i) = \cup_{i=1}^p (G_{\mathbb{C}} \cdot S_i) \subset M^{(H)}_{(\mathfrak{h}')}$. If $\phi(M^{(H)}_{(\mathfrak{h}')})$ lies on another face $\mathcal{F}_{p+1}$ whose closure does not contain 0, then $G_{\mathbb{C}} \cdot S_{p+1}$ (where S_{p+1} is taken similarly as the S_i's) should emerge into $\cup_{i=1}^p G_{\mathbb{C}} \cdot S_i$ to converge to $\phi^{-1}(0)$. This proves $M^{(H)}_{(\mathfrak{h}')} = \cup_{i\in I}(G_{\mathbb{C}} \cdot S_i)$. □

By this lemma, if two faces $\mathcal{F}_i$ and $\mathcal{F}_j$ where $i, j \in I$ contain a one dimensional less face $\mathcal{F}_k$ in their common closure, then $G_{\mathbb{C}} \cdot S_i \cap G_{\mathbb{C}} \cdot S_j = G_{\mathbb{C}} \cdot S_k$.

Remark 3. In the above lemma, generally we cannot get all $G_{\mathbb{C}} \cdot S_i$ from $G_{\mathbb{C}} \cdot S_k$ by the flow lines of JX^ξ, where $\xi \in (m)$. Some orbits in $G_{\mathbb{C}} \cdot S_i$ may converge to more singular orbits in $\phi^{-1}(G \cdot \mathcal{F}_k)$.

Since 0 is in the closure of each $\mathcal{F}_i$, dim $(\mathcal{F}_i) > 0$ for each $i = 1, \ldots, p$.

So we have proved

Lemma 10. *We can decompose $F_\infty^{-1}(Z_{(H)})$ into a (finite) disjoint union $F_\infty^{-1}(Z_{(H)}) = G_{\mathbb{C}} \cdot Z_{(H)} \bigcup_{\mathfrak{h}'} M^{(H)}_{(\mathfrak{h}')}$, where $\bigcup_{\mathfrak{h}'} M^{(H)}_{(\mathfrak{h}')} = \emptyset$, or, each $M^{(H)}_{(\mathfrak{h}')}$ can be written as in Lemma 8 or in Lemma 9. In the second case, for any $i = 0, 1 \ldots, p$, we may choose $a_i' \neq 0$ on $\mathcal{F}_i$ different from a_i and choose $S_i' \subseteqq \{x \in \phi^{-1}(G \cdot a_i') : x$ has isotropy Lie algebra type $(\mathfrak{h}')\}$ and we have $G_{\mathbb{C}} \cdot S_i = G_{\mathbb{C}} \cdot S_i'$.*

Definition 2. Let $n_{(H)}$ be the complex dimension of $G_{\mathbb{C}} \cdot Z_{(H)}$, and let $n^{(H)}_{(\mathfrak{h}')}$ be the complex dimension of $M^{(H)}_{(\mathfrak{h}')}$. Take $s \in \mathcal{H}(M, L^{\otimes k})^G$. Define

$$I_k^{Z_{(H)}} = (k/2\pi)^{n_{(H)}/2} \int_{G_{\mathbb{C}} \cdot Z_{(H)}} |s|^2 \mathrm{dvol}(G_{\mathbb{C}} \cdot Z_{(H)}),$$

and,

$$II_k^{Z_{(H)}} = \sum_{\mathfrak{h}'} (k/2\pi)^{n^{(H)}_{(\mathfrak{h}')}/2} \int_{M^{(H)}_{(\mathfrak{h}')}} |s|^2 \mathrm{dvol}(M^{(H)}_{(\mathfrak{h}')}).$$

Define

$$\int_{F_\infty^{-1}(Z_{(H)})} |s|^2 \mathrm{dvol}(F_\infty^{-1}(Z_{(H)})) = I_k^{Z_{(H)}} + II_k^{Z_{(H)}}.$$

Lemma 11. *Let $s \in \mathcal{H}(M, L^{\otimes k})^G$. Then*

(a) $I_k^{Z_{(H)}} = (k/2\pi)^{d_{\mathcal{S}_{(H)}}/2} \int_{\mathcal{S}_{(H)}} |A'_k s|^2([x]) I_k^{\mathcal{S}_{(H)}}([x]) \epsilon_{\hat{\omega}_{\mathcal{S}_{(H)}}}$, *where*

$$I_k^{\mathcal{S}_{(H)}}([x]) = 1, \quad \text{if } H = G; \quad \text{and}$$

$$I_k^{\mathcal{S}_{(H)}}([x]) = \mathrm{vol}(G \cdot x)(k/2\pi)^{d_{G/H}/2} \int_{\mathfrak{m}} \tau(\xi, x) \exp\left\{ -\int_0^1 2k \phi_\xi(\mathrm{e}^{\mathrm{i}t\xi} \cdot x) \mathrm{d}t \right\} \mathrm{dvol}(\mathfrak{m}),$$

if $H \neq G$. Here, $\mathfrak{m}$ denotes the orthogonal complement of $\mathfrak{h} = Lie(H)$ in $\mathfrak{g}$, and x is a point with stabilizer group H.

(b)

$$II_k^{Z_{(H)}} = 0,$$

or

$$II_k^{Z_{(H)}} = \sum_{\mathfrak{h}'} (k/2\pi)^{n^{(H)}_{(\mathfrak{h}')}/2} \left(\sum_i \pm \int_{S_i} |s|^2 (g \cdot u) \mathrm{dvol}(S_i) \int_{\mathfrak{m}'} \tau(\zeta, u) \exp\left\{ -2k \int_0^1 \phi_\zeta(\mathrm{e}^{it\zeta} \cdot u) \right\} \mathrm{dvol}(\mathfrak{m}') \right),$$

where the second sum is over some subset of indices of i occurring in Lemma 8 or in Lemma 9, $\mathfrak{m}'$ is the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{g}$, and the points $u \in S_i$ are of the isotropy Lie algebra $\mathfrak{h}'$.

Proof. We will drop the subscripts and superscripts in $I_k^{Z_{(H)}}$ and $II_k^{Z_{(H)}}$ and simply write I and II.

(a) If $H = G$, then $G_{\mathbb{C}} \cdot Z_G = Z_G = \mathcal{S}_G$. So

$$\begin{aligned} I &= (k/2\pi)^{n_{(H)}/2} \int_{G_{\mathbb{C}} \cdot Z_G} |s|^2 \mathrm{dvol}(G_{\mathbb{C}} \cdot Z_G) = (k/2\pi)^{n_{(H)}/2} \int_{Z_G} |s|^2(x) \mathrm{dvol}(Z_G) \\ &= (k/2\pi)^{d_{\mathcal{S}_G}/2} \int_{\mathcal{S}_G} |A'_k s|^2([x]) \epsilon_{\hat{\omega}_{\mathcal{S}_G}}. \end{aligned}$$

If $H \neq G$, then by the coarea formula, the formula (9), and Theorem 8(a), we have

$$I = (k/2\pi)^{n_{(H)}/2} \int_{Z_{(H)}} |s|^2(x') \mathrm{dvol}(Z_{(H)}) \int_{\mathfrak{m}} \tau(\xi, g^{-1}x') \exp\left\{ -2k \int_0^1 \phi_{Ad(g)\xi}(\mathrm{e}^{\mathrm{i}t Ad(g)\xi} \cdot x') \right\} \mathrm{dvol}(\mathfrak{m}),$$

where x' is any point in $Z_{(H)}$ with stabilizer group gHg^{-1} (for some g).

By the G-invariance of the function τ, and by G-equivariance of the moment map ϕ, we have

$$\begin{aligned} &\int_{\mathfrak{m}} \tau(\xi, g^{-1}x') \exp\left\{ -2k \int_0^1 \phi_{Ad(g)\xi}(\mathrm{e}^{\mathrm{i}t Ad(g)\xi} \cdot x') \right\} \mathrm{dvol}(\mathfrak{m}) \\ &\quad = \int_{\mathfrak{m}} \tau(\xi, x) \exp\left\{ -2k \int_0^1 \phi_\xi(\mathrm{e}^{\mathrm{i}t\xi} \cdot x) \right\} \mathrm{dvol}(\mathfrak{m}), \end{aligned}$$

where $x = g^{-1}x'$ has stabilizer H.

Using the fact that $\mathrm{dvol}(Z_{(H)}) = \mathrm{dvol}(G \cdot x) \wedge \pi^* \mathrm{dvol}(\mathcal{S}_{(H)})$, the integral

$$I = (k/2\pi)^{(n_{(H)} - d_{G/H})/2} \int_{\mathcal{S}_{(H)}} |A'_k s|^2([x]) \mathrm{dvol}(\mathcal{S}_{(H)}) \mathrm{vol}(G \cdot x)(k/2\pi)^{d_{G/H}/2}$$

$$\int_{\mathfrak{m}} \tau(\xi, x) \exp\left\{-2k \int_{\gamma_\xi} \phi_\xi\right\} \mathrm{dvol}(\mathfrak{m}) = (k/2\pi)^{d_{\mathcal{S}_{(H)}}/2} \int_{\mathcal{S}_{(H)}} |A'_k s|^2([x]) I_k^{\mathcal{S}_{(H)}}([x]) \epsilon_{\hat{\omega}_{\mathcal{S}_{(H)}}},$$

where

$$I_k^{\mathcal{S}_{(H)}}([x]) = \mathrm{vol}(G \cdot x)(k/2\pi)^{d_{G/H}/2} \int_{\mathfrak{m}} \tau(\xi, x) \exp\left\{-2k \int_0^1 \phi_\xi(\mathrm{e}^{\mathrm{i}t\xi} \cdot x)\right\} \mathrm{dvol}(\mathfrak{m}),$$

with x being taken as a point (on the orbit $G \cdot x$) with stabilizer group exactly H.

(b) If $\bigcup_{\mathfrak{h}'} M^{(H)}_{(\mathfrak{h}')} = \emptyset$, then $II = 0$.

Otherwise, let us only consider one summand for the first summation in II. The others follow similarly. So, we assume

$$II = (k/2\pi)^{n^{(H)}_{(\mathfrak{h}')}/2} \int_{M^{(H)}_{(\mathfrak{h}')}} |s|^2 \mathrm{dvol}(M^{(H)}_{(\mathfrak{h}')}).$$

By Lemma 8 or Lemma 9, we can compute this integral over one set $G_{\mathbb{C}} \cdot S_0$, or we can compute it over a finite union $G_{\mathbb{C}} \cdot S_{i \in I}$ and possibly subtract some integrals over some mutual intersections which have similar forms (if a mutual intersection has lower dimension, then we do not subtract). So we only need to write one such integral in the stated form.

Using the coarea formula, the formula (9) and Theorem 8(a) on the space $G_{\mathbb{C}} \cdot S_i$, we have

$$\int_{G_{\mathbb{C}} \cdot S_i} |s|^2 \mathrm{dvol}(G_{\mathbb{C}} \cdot S_i)$$
$$= \int_{S_i} |s|^2(u') \mathrm{dvol}(S_i) \int_{\mathfrak{m}'} \tau(\zeta, g^{-1}u') \exp\left\{-2k \int_0^1 \phi_{Ad(g)\zeta}(\mathrm{e}^{itAd(g)\zeta} \cdot u')\right\} \mathrm{dvol}(\mathfrak{m}'),$$

where $u' \in S_i$ is any point with isotropy Lie algebra $Ad(g)\mathfrak{h}'$ (for some g).

For the same reason as in (a), we have

$$\int_{\mathfrak{m}'} \tau(\zeta, g^{-1}u') \exp\left\{-2k \int_0^1 \phi_{Ad(g)\zeta}(\mathrm{e}^{itAd(g)\zeta} \cdot u')\right\} \mathrm{dvol}(\mathfrak{m}')$$
$$= \int_{\mathfrak{m}'} \tau(\zeta, u) \exp\left\{-2k \int_0^1 \phi_\zeta(\mathrm{e}^{it\zeta} \cdot u)\right\} \mathrm{dvol}(\mathfrak{m}'),$$

where $u = g^{-1}u'$ has isotropy Lie algebra $\mathfrak{h}'$. □

Definition 3. We use the same notations as those in Definition 2. Take $r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. Define

$$\tilde{I}_k^{Z_{(H)}} = (k/2\pi)^{n_{(H)}/2} \int_{G_{\mathbb{C}} \cdot Z_{(H)}} |r|^2 \mathrm{dvol}(G_{\mathbb{C}} \cdot Z_{(H)}),$$

and

$$\widetilde{II}_k^{Z_{(H)}} = \sum_{\mathfrak{h}'} (k/2\pi)^{n^{(H)}_{(\mathfrak{h}')}/2} \int_{M^{(H)}_{(\mathfrak{h}')}} |r|^2 \mathrm{dvol}(M^{(H)}_{(\mathfrak{h}')}).$$

Define

$$\int_{F_\infty^{-1}(Z_{(H)})} |r|^2 \mathrm{dvol}(F_\infty^{-1}(Z_{(H)})) = \tilde{I}_k^{Z_{(H)}} + \widetilde{II}_k^{Z_{(H)}}.$$

Lemma 12. *Let* $r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. *Then*

(a) $\tilde{I}_k^{Z(H)} = (k/2\pi)^{d_{\mathcal{S}(H)}/2} \int_{\mathcal{S}(H)} |B_k' r|^2([x]) J_k^{\mathcal{S}(H)}([x]) \epsilon_{\hat{\omega}_{\mathcal{S}(H)}}$, *where*

$$J_k^{\mathcal{S}(H)}([x]) = 1, \text{ if } \quad H = G; \quad \text{and} \quad J_k^{\mathcal{S}(H)}([x]) = (k/2\pi)^{d_{G/H}/2} 2^{d_{G/H}/2} \cdot \int_{\mathfrak{m}} \tau(\xi, x) \exp\left\{ -\int_0^1 \left(2k\phi_\xi(\mathrm{e}^{\mathrm{i}t\xi} \cdot x) + \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega}(\mathrm{e}^{\mathrm{i}t\xi} \cdot x) \right) \right\} \mathrm{dvol}(\mathfrak{m}),$$

if $H \neq G$. *Here,* $\mathfrak{m}$ *denotes the orthogonal complement of* $\mathfrak{h} = Lie(H)$ *in* $\mathfrak{g}$, *and* x *is a point with stabilizer group* H.

(b)

$$\widetilde{II}_k^{Z(H)} = 0,$$

or

$$\widetilde{II}_k^{Z(H)} = \sum_{\mathfrak{h}'} (k/2\pi)^{n^{(H)}_{(\mathfrak{h}')}/2} \left(\sum_i \pm \int_{S_i} |r|^2 (g \cdot u) \mathrm{dvol}(S_i) \int_{\mathfrak{m}'} \tau(\zeta, u) \times \exp\left\{ -\int_0^1 \left(2k\phi_\zeta(\mathrm{e}^{it\zeta} \cdot u) + \frac{\mathcal{L}_{JX^\zeta}\epsilon_\omega}{2\epsilon_\omega}(\mathrm{e}^{it\zeta} \cdot u) \right) \right\} \mathrm{dvol}(\mathfrak{m}') \right),$$

where the second sum is over some subset of indices of i *occuring in Lemma 8 or Lemma 9,* $\mathfrak{m}'$ *is the orthogonal complement of* $\mathfrak{h}'$ *in* $\mathfrak{g}$, *and the points* $u \in S_i$ *are of isotropy Lie algebra* $\mathfrak{h}'$.

Proof. (a) The proof is similar to the proof of (a) of Lemma 11, but we will use Theorem 10. We will drop the subscript and superscript in $\tilde{I}_k^{Z(H)}$ and simply write $\tilde{I}$.

If $H = G$, then $G_{\mathbb{C}} \cdot Z_G = Z_G = \mathcal{S}_G$. Then

$$\begin{aligned} \tilde{I} &= (k/2\pi)^{n_G/2} \int_{G_{\mathbb{C}} \cdot Z_G} |r|^2 \mathrm{dvol}(G_{\mathbb{C}} \cdot Z_G) \\ &= (k/2\pi)^{n_G/2} \int_{Z_G} |r|^2(x) \mathrm{dvol}(Z_G) \\ &= (k/2\pi)^{d_{\mathcal{S}_G}/2} \int_{\mathcal{S}_G} |B_k' r|^2([x]) \epsilon_{\hat{\omega}_{\mathcal{S}_G}} \end{aligned}$$

by Theorem 10.

If $H \neq G$, then by the coarea formula, the formula (9), Theorem 8(b), and by a G-invariance argument as in the proof of Lemma 11, we have

$$\tilde{I} = (k/2\pi)^{n_{(H)}/2} \int_{Z_{(H)}} |r|^2 (g \cdot x) \mathrm{dvol}(Z_{(H)}) \int_{\mathfrak{m}} \tau(\xi, x) \exp\left\{ -\int_{\gamma_\xi} \left(2k\phi_\xi + \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega} \right) \right\} \mathrm{dvol}(\mathfrak{m}).$$

By Theorem 10,

$$\begin{aligned} \tilde{I} &= (k/2\pi)^{d_{\mathcal{S}(H)}/2} \int_{\mathcal{S}(H)} |B_k' r|^2([x]) \mathrm{dvol}(\mathcal{S}_{(H)}) 2^{d_{G/H}/2} (k/2\pi)^{d_{G/H}/2} \int_{\mathfrak{m}} \tau(\xi, x) \\ &\quad \times \exp\left\{ -\int_{\gamma_\xi} \left(2k\phi_\xi + \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega} \right) \right\} \mathrm{dvol}(\mathfrak{m}) \\ &= (k/2\pi)^{d_{\mathcal{S}(H)}/2} \int_{\mathcal{S}(H)} |B_k' r|^2([x]) J_k^{\mathcal{S}(H)}([x]) \epsilon_{\hat{\omega}_{\mathcal{S}(H)}}, \end{aligned}$$

where

$$J_k^{\mathcal{S}(H)}([x]) = (k/2\pi)^{d_{G/H}/2} 2^{d_{G/H}/2} \int_{\mathfrak{m}} \tau(\xi, x) \exp\left\{ -\int_{\gamma_\xi} \left(2k\phi_\xi + \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega} \right) \right\} \mathrm{dvol}(\mathfrak{m}).$$

(b) Similar to the proof of (b) of Lemma 11. We omit it. □

Now, we come to our main result of this section:

Theorem 11. (a) *Let* $s \in \mathcal{H}(M, L^{\otimes k})^G$. *Then*

$$\sum_{Z_{(H)}} \int_{F_\infty^{-1}(Z_{(H)})} |s|^2 \mathrm{dvol}(F_\infty^{-1}(Z_{(H)})) = \sum_{\mathcal{S}_{(H)}} (k/2\pi)^{d_{\mathcal{S}_{(H)}}/2} \int_{\mathcal{S}_{(H)}} |A'_k s|^2([x]) I_k^{\mathcal{S}_{(H)}}([x]) \epsilon_{\hat{\omega}_{\mathcal{S}_{(H)}}} + \sum_{Z_{(H)}} II_k^{Z_{(H)}},$$

where $I_k^{\mathcal{S}_{(H)}}([x])$ *is as in Lemma* 11(a), *and each* $II_k^{Z_{(H)}}$ *is as in Lemma* 11(b).

In particular, the above is true for each individual summand with respect to (H).

(b) *Let* $r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. *Then*

$$\sum_{Z_{(H)}} \int_{F_\infty^{-1}(Z_{(H)})} |r|^2 \mathrm{dvol}(F_\infty^{-1}(Z_{(H)})) = \sum_{\mathcal{S}_{(H)}} (k/2\pi)^{d_{\mathcal{S}_{(H)}}/2} \int_{\mathcal{S}_{(H)}} |B'_k r|^2([x]) J_k^{\mathcal{S}_{(H)}}([x]) \epsilon_{\hat{\omega}_{\mathcal{S}_{(H)}}} + \sum_{Z_{(H)}} \widetilde{II}_k^{Z_{(H)}},$$

where $J_k^{\mathcal{S}_{(H)}}([x])$ *is as in Lemma* 12(a), *and each* $\widetilde{II}_k^{Z_{(H)}}$ *is as in Lemma* 12(b).

In particular, the above is true for each individual summand with respect to (H).

Proof. Lemmas 11 and 12 proved the statements for the individual summands. The statements for the sums follow from these lemmas by taking the sum of the individual terms. □

The asymptotic properties of $I_k^{\mathcal{S}_{(H)}}([x])$, of $J_k^{\mathcal{S}_{(H)}}([x])$, of $II_k^{Z_{(H)}}$ and of $\widetilde{II}_k^{Z_{(H)}}$ will be studied in the next section (see Theorem 12).

12. Asymptotics

Our main result of this section is

Theorem 12. (a) *The densities* $I_k^{\mathcal{S}_{(H)}}$ *and* $J_k^{\mathcal{S}_{(H)}}$ *for* $H \neq G$ *satisfy*

$$\lim_{k\to\infty} I_k^{\mathcal{S}_{(H)}}([x]) = 2^{-d_{G/H}/2}\mathrm{vol}(G \cdot x),$$

and

$$\lim_{k\to\infty} J_k^{\mathcal{S}_{(H)}}([x]) = 1.$$

The limits are uniform for $[x] \in Z_{(H)}/G$.

(b) *If* $II_k^{Z_{(H)}} \neq 0$ *and* $\widetilde{II}_k^{Z_{(H)}} \neq 0$, *then they satisfy*

$$\lim_{k\to\infty} II_k^{Z_{(H)}} = 0,$$

and

$$\lim_{k\to\infty} \widetilde{II}_k^{Z_{(H)}} = 0.$$

The proof of this theorem will be given in Section 12.3.

12.1. Growth estimates

In Lemmas 11 and 12, in the expressions of $I_k^{\mathcal{S}_{(H)}}$, or of $J_k^{\mathcal{S}_{(H)}}$ (for $H \neq G$), or in the summands of $II_k^{Z_{(H)}}$ or $\widetilde{II}_k^{Z_{(H)}}$, we had the following types of integrals

$$\int_{\mathfrak{m}} \tau(\xi, x) \exp\left\{-2k \int_{\gamma_\xi} \phi_\xi\right\} \mathrm{dvol}(\mathfrak{m})$$

and

$$\int_{\mathfrak{m}} \tau(\xi, x) \exp\left\{-\int_{\gamma_\xi} \left(2k\phi_\xi + \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega}\right)\right\} \mathrm{dvol}(\mathfrak{m}),$$

where $x \in S$ with $S = Z_{(H)}$ or $S = S_i$ for some S_i as in Lemma 8 or in Lemma 9, $\xi \in \mathfrak{m}$, and $\gamma_\xi = \mathrm{e}^{\mathrm{i}t\xi} \cdot x$, $t \in [0, 1]$.

Remark 4. In this and the next subsections, for simplicity, we will only use $\mathfrak{m}$ to denote the orthogonal complement of $\mathfrak{h}$ or of $\mathfrak{h}'$ in $\mathfrak{g}$, as we did in Formula (9).

Theorem 13. *Consider $G_{\mathbb{C}} \cdot S$, where $S = Z_{(H)}$ or $S = S_i$ for an S_i as in Lemma 8 or in Lemma 9. There exist constants b, and $D > 0$ such that for all $[x] \in S/G$ (the integral is a function of $[x]$), and for all R and k sufficiently large,*

$$\int_{\mathfrak{m}-B_R(0)} \tau(\xi, x) \exp\left\{-2k \int_{\gamma_\xi} \phi_\xi\right\} \mathrm{dvol}(\mathfrak{m}) \le b\mathrm{e}^{-RDk},$$

where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ or of $\mathfrak{h}'$ in $\mathfrak{g}$, and $B_R(0)$ is a ball in $\mathfrak{m}$ of radius R centered at 0.

Since we can find a uniform bound for $-\frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega}$ on M, the above inequality is also true for the integral $\int_{\mathfrak{m}-B_R(0)} \tau(\xi, x) \exp\{-\int_{\gamma_\xi}(2k\phi_\xi + \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega})\}\mathrm{dvol}(\mathfrak{m})$.

The proof of this theorem relies on the following two lemmas.

Lemma 13. *Consider $G_{\mathbb{C}} \cdot S$, where $S = Z_{(H)}$ or $S = S_i$ for an S_i as in Lemma 8 or in Lemma 9. For any $t_0 > 0$, there exists $C > 0$ such that for all $t > t_0$,*

$$\exp\left\{-\int_{\gamma_{t\hat{\xi}}} 2k\phi_{t\hat{\xi}}\right\} \le \mathrm{e}^{-2ktC}$$

uniformly on S/G, where $\hat{\xi} \in \mathfrak{m}$ with $|\hat{\xi}| = 1$.

Proof. By definition, $\int_{\gamma_{t\hat{\xi}}} \phi_{t\hat{\xi}} = \int_0^1 \langle \phi(\mathrm{e}^{\mathrm{i}\tau t\hat{\xi}} \cdot x), t\hat{\xi}\rangle \mathrm{d}\tau = t\int_0^1 \langle \phi(\mathrm{e}^{\mathrm{i}\tau t\hat{\xi}} \cdot x), \hat{\xi}\rangle \mathrm{d}\tau$. Hence, we need to find a positive lower bound for the function $f_t(\hat{\xi}, x) = \int_0^1 \phi_{\hat{\xi}}(\mathrm{e}^{\mathrm{i}\tau t\hat{\xi}} \cdot x)\mathrm{d}\tau$ when t is sufficiently large. We prove the lemma for the case $S = Z_{(H)}$. The argument applies to other cases. Since f_t is G-invariant, on each G-orbit, we only need to consider a particular point x which has isotropy Lie algebra exactly $\mathfrak{h}$. So we take $S^{\mathfrak{h}} \subset S$ to be the set of such points. First, fix $\hat{\xi} \in \mathfrak{m}$ with $|\hat{\xi}| = 1$, and consider the $\hat{\xi}$-moment map $\phi_{\hat{\xi}}$. Then, $\phi_{\hat{\xi}}(S^{\mathfrak{h}}) =$ constant. For $x \in S^{\mathfrak{h}}$, $f_t(\hat{\xi}, x)$ for any $t > 0$ is strictly increasing since $\mathrm{e}^{\mathrm{i}\tau t\hat{\xi}} \cdot x$ is the gradient line of $\phi_{t\hat{\xi}}$ and $JX^{t\hat{\xi}}(x) \neq 0$. If $S^{\mathfrak{h}}$ is compact, we can find a positive lower bound $C_{\hat{\xi}}$ for $f_t(\hat{\xi}, x)$ for all points in $S^{\mathfrak{h}}$ and for all $t > t_0$ for any chosen $t_0 > 0$. If $S^{\mathfrak{h}}$ is not compact, we do the following. Consider a nearby regular value $a > 0$ of $\phi_{\hat{\xi}}$. For $y \in \phi_{\hat{\xi}}^{-1}(a)$, consider the function $f_t'(\hat{\xi}, y) = \int_{-\epsilon}^{1-\epsilon} \phi_{\hat{\xi}}(\mathrm{e}^{\mathrm{i}\tau t\hat{\xi}} \cdot y)\mathrm{d}\tau$, where ϵ is a small number. By choosing a properly, for each x in $S^{\mathfrak{h}}$, there exists $y \in \phi_{\hat{\xi}}^{-1}(a)$, such that $f_t'(\hat{\xi}, y) = f_t(\hat{\xi}, x)$ (since the x's are not fixed by the circle action generated by $\hat{\xi}$, this can be achieved). We choose the positive minimum of f_t' on its compact domain $\phi_{\hat{\xi}}^{-1}(a)$ as $C_{\hat{\xi}}$ (the positivity of $C_{\hat{\xi}}$ is due to a being a regular value). So, for each $\hat{\xi} \in \mathfrak{m}$ with $|\hat{\xi}| = 1$, there exists $C_{\hat{\xi}} > 0$, such that $f_t(\hat{\xi}, x) = \int_0^1 \phi_{\hat{\xi}}(\mathrm{e}^{\mathrm{i}\tau t\hat{\xi}} \cdot x)\mathrm{d}\tau \ge C_{\hat{\xi}}$ for all $[x] \in S/G$. By the compactness of the set $\{\hat{\xi} \in \mathfrak{m}, |\hat{\xi}| = 1\}$, and by the continuous dependence of f_t on $\hat{\xi}$, we can find a positive constant C such that $f_t \ge C$ uniformly for all $[x] \in S/G$ and for all $\hat{\xi} \in \mathfrak{m}$ with $|\hat{\xi}| = 1$.

For the proof of other S's, we replace the above $S^{\mathfrak{h}}$ by $S^{\mathfrak{h}'} \cap \phi^{-1}(a_i)$ (recall that $0 \neq a_i \in \mathcal{F}_i$) so that $\phi_{\hat{\xi}}(S^{\mathfrak{h}'} \cap \phi^{-1}(a_i)) =$ constant, noticing the fact that $S^{\mathfrak{h}'} \cap \phi^{-1}(a_i)$ has all the representatives of S/G. □

Lemma 14. *Consider $G_{\mathbb{C}} \cdot S$, where $S = Z_{(H)}$ or $S = S_i$ for an S_i as in Lemma 8 or in Lemma 9. There exist constants a and $b > 0$ such that for all $t > 0$*

$$\tau(t\hat{\xi}, x) \leq bt^{-m}\mathrm{e}^{at}$$

uniformly on S/G, where $\hat{\xi} \in \mathfrak{m}$ with $|\hat{\xi}| = 1$, and m is the dimension of $\mathfrak{m}$.

Proof. The manifold $G_{\mathbb{C}} \cdot S$ is a complex submanifold of M. Since M can be embedded into projective spaces, $G_{\mathbb{C}} \cdot S$ is a complex submanifold of projective spaces. The proof of Lemma 5.7 in [9] applies. (The proof of Lemma 5.7 in [9] does not need the domain of x to be compact, but it uses the fact that the domain of $\hat{\xi}$ is compact.) □

Once we have the above two lemmas, using polar coordinates, we can prove Theorem 13. One may refer to the proof of Theorem 5.5 in [9].

12.2. Approximation

Lemma 15. *The function $\tau(\xi, x)$ equals $vol(G \cdot x)$ on S, where $S = Z_{(H)}$ or $S = S_i$ for an S_i as in Lemma 8 or in Lemma 9.*

Proof. We prove the lemma for the case $S = S_i$ for some i. The proof for the other S's is similar. Consider the complex submanifold $G_{\mathbb{C}} \cdot S$. We take $S^{\mathfrak{h}'} \subset S$, the set of points with isotropy Lie algebra exactly $\mathfrak{h}'$. Let $\tilde{S} = S^{\mathfrak{h}'} \cap \phi^{-1}(a_i)$. Then $\tilde{S}$ contains all the representatives of S/G. Since $\tau(\xi, x)$ is G-invariant, we only need to consider the value $\tau(0, x)$ with $x \in \tilde{S}$. So we only consider Formula (9) on $\mathrm{e}^{i\mathfrak{m}} \cdot \tilde{S}$. Consider the submanifold $\mathrm{e}^{i\mathfrak{m}} \cdot \tilde{S}$. At each point $x \in \tilde{S}$, the B-orthogonal complement of $T_x\tilde{S}$ in $T_x(\mathrm{e}^{i\mathfrak{m}} \cdot \tilde{S})$ is exactly the linear span of the vectors JX^{ξ} with $\xi \in \mathfrak{m}$: for any JX^{ξ} with $\xi \in \mathfrak{m}$ and any vector $v \in T_x\tilde{S}$, we have $B(JX^{\xi}, v)_x = \omega(v, X^{\xi})_x = v(\phi_{\xi})_x = 0$, since ϕ_{ξ} takes constant value on $\tilde{S}$. So B is block diagonalizable at x on the submanifold $\mathrm{e}^{i\mathfrak{m}} \cdot \tilde{S}$, and

$$\mathrm{dvol}(\mathrm{e}^{i\mathfrak{m}} \cdot \tilde{S})_x = \sqrt{\det B(JX^{\xi_i}, JX^{\xi_j})_x}\, \mathrm{dvol}(\mathfrak{m}) \wedge \mathrm{dvol}(\tilde{S})_x$$

with $\xi_i, \xi_j \in \mathfrak{m}$. By Lemma 3, $\sqrt{\det B(JX^{\xi_i}, JX^{\xi_j})_x} = \mathrm{vol}(G \cdot x)$. □

The result of the above lemma will be used in the proof of the following lemma.

Lemma 16. *Consider $G_{\mathbb{C}} \cdot S$, where $S = Z_{(H)}$ or $S = S_i$ for an S_i as in Lemma 8 or in Lemma 9. Define*

$$I_{k,R}([x]) = (k/2\pi)^{m/2} \int_{B_R(0)} \tau(\xi, x)\mathrm{e}^{-kf(\xi,x)}\mathrm{dvol}(\mathfrak{m}),$$

where $f(\xi, x) = 2\int_0^1 \phi_{\xi}(\mathrm{e}^{\mathrm{i}t\xi} \cdot x)\mathrm{d}t$ at a point $x \in S$ with isotropy Lie algebra $\mathfrak{h}$ or $\mathfrak{h}'$, $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ or of $\mathfrak{h}'$ in $\mathfrak{g}$, and $m = \dim(\mathfrak{m})$.

Then there exists some $R > 0$ such that

$$\lim_{k\to\infty} |I_{k,R}([x]) - 2^{-m/2}| = 0$$

uniformly on S/G.

Proof. We will prove the lemma for the case $S = Z_{(H)}$. The other cases follow similarly. We refer to the proof of Lemma 5.10 in [9]. By Theorem 8, the function $f(\xi, x)$ is a G-invariant Morse–Bott function on $G_{\mathbb{C}} \cdot Z_{(H)}$ with $0 \times Z_{(H)}$ being a minimum. By the Morse–Bott lemma, for each point $x \in Z_{(H)}$, there exists a neighborhood of this point on which $f(\xi, x)$ can be written as a quardratic function. If $Z_{(H)}$ is compact, we can choose the smallest positive radius of the (finitely many) neighborhoods as R. If $Z_{(H)}$ is not compact, note that if $Z_{(K)}$ is in the closure of $Z_{(H)}$, then (up to conjugacy) the orthogonal complement $\mathfrak{m}'$ of $Lie(K)$ is a linear subspace of the orthogonal complement $\mathfrak{m}$ of $Lie(H)$. Because of this property, for the function $f(\xi, x)$ on $G_{\mathbb{C}} \cdot Z_{(K)}$ we may assume that the neighborhoods of the points x's $\in Z_{(K)}$ overlap the strata $Z_{(H)}$'s whose closures contain $Z_{(K)}$. So, we can use the compactness of $\phi^{-1}(0)$ to have finitely many neighborhoods, and therefore choose the smallest R for all the strata $Z_{(H)} \subset \phi^{-1}(0)$. Once R is chosen, on each $G_{\mathbb{C}} \cdot Z_{(H)}$, follow the arguments of the proof of Lemma 5.10 in [9]. In the proof of Lemma 5.10 in [9], there are some estimates on the bounds of the absolute value of some continuous functions of

$x \in Z_{(H)}$ which involve certain integrals of the derivative of $\tau(\xi, x)$ in the direction of ξ (the constants Q_1 and Q_2). If $Z_{(H)}$ is not compact, the formula on τ in (9) of Section 11.1 should continuously transform from higher dimensional strata $Z_{(H)}$ of $\phi^{-1}(0)$ to lower dimensional ones. This should allow us to extend continuously the above continuous functions to the closure of $Z_{(H)}$ in $\phi^{-1}(0)$ and take the maximum of the absolute values. (The constant Q_3 in the proof of Lemma 5.10 in [9], is $2^{m/2}$ in our case.) □

Lemma 17. *Consider $G_{\mathbb{C}} \cdot S$, where $S = Z_{(H)}$ or $S = S_i$ for an S_i as in Lemma 8 or in Lemma 9. Define*

$$J_{k,R}([x]) = (k/2\pi)^{m/2} 2^{m/2} \int_{B_R(0)} \tau(\xi, x) e^{-kf(\xi,x)} \exp\left\{-\int_{\gamma_\xi} \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega}\right\} \mathrm{dvol}(\mathfrak{m}).$$

Then, there exists $R > 0$ such that

$$\lim_{k\to\infty} |J_{k,R}([x]) - 1| = 0$$

uniformly on S/G.

Proof. In the proof of Lemma 16, replace $\tau(\xi, x)$ by $\tau(\xi, x) \exp\{-\int_{\gamma_\xi} \frac{\mathcal{L}_{JX^\xi}\epsilon_\omega}{2\epsilon_\omega}\}$; just notice that the exponent is 0 when $\xi = 0$. □

12.3. Proof of Theorem 12

Proof. (a) We write $I_k^{\mathcal{S}_{(H)}}$ as the sum of an integral over $B_R(0)$ and an integral over the complement of $B_R(0)$. The result follows from Lemma 16 and Theorem 13. The proof for $J_k^{\mathcal{S}_{(H)}}$ is similar but using Lemma 17 and Theorem 13.

(b) We assume that $II_k^{Z_{(H)}} \neq 0$ and $\widetilde{II}_k^{Z_{(H)}} \neq 0$.

Now we prove $\lim_{k\to\infty} II_k^{Z_{(H)}} = 0$. Since we have a finite summation in the expression of $II_k^{Z_{(H)}}$, we only need to prove that each summand goes to 0 when $k \to \infty$. We will simply write $G_{\mathbb{C}} \cdot S_i$ as $G_{\mathbb{C}} \cdot S$. By Theorem 13 and Lemma 16, there exists $K_0 > 0$, such that when $k > K_0$,

$$\begin{aligned}\int_{\mathfrak{m}'} \tau(\zeta, u) \exp\left\{-2k \int_{\gamma_\zeta} \phi_\zeta\right\} \mathrm{dvol}(\mathfrak{m}') &\leq b e^{-RDk} + 2(k/2\pi)^{-m'/2} 2^{-m'/2} \\ &\leq b e^{-RDk} + (k/2\pi)^{-m'/2} b'.\end{aligned}$$

Now, let us consider the term $\int_S |s|^2(u)\mathrm{dvol}(S)$. By Lemma 10, we can take $a' \in \mathcal{F}$ and take $S' \subset \phi^{-1}(G \cdot a')$ such that S can be reached from S' by following the flow lines of the vector fields JX^ζ, where $\zeta \in (\mathfrak{m}')$. We use Theorem 8(a) to express $|s|^2(u)$ in terms of $|s|^2(u')$ and we use the arguments in the proof of Lemma 13 to find a constant $C' > 0$ such that $|s|^2(u) \leq |s|^2(u') e^{-kC'}$ for all $u' \in S'$. Now, since M is compact, $|s|^2(u')$ is bounded. The volume of S is also bounded. So $\int_S |s|^2(u)\mathrm{dvol}(S) \leq C'' e^{-kC'}$ for some constant C''.

So, for each summand in $II_k^{Z_{(H)}}$, there exist $K_0 > 0$ and constants C, C', b, b', R, D with C', R and D positive such that when $k > K_0$, the summand

$$\leq (k/2\pi)^{n_{(\mathfrak{h})}^{(H)}/2} C e^{-kC'} (b e^{-RDk} + (k/2\pi)^{-m'/2} b').$$

Therefore $\lim_{k\to\infty} II_k^{Z_{(H)}} = 0$.

The proof for the statement about $\widetilde{II}_k^{Z_{(H)}}$ is similar. □

13. Asymptotic unitarity

Now, it comes to the definition of the inner products on $\mathcal{H}(M, L^{\otimes k})^G$ and on $\mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. Recall that M^{ss} is open and dense in M, and

$$M^{ss} = \bigcup_{(H)} F_\infty^{-1}(Z_{(H)}).$$

There is an open and dense set $F_\infty^{-1}(Z_{(H)})$ for some H in M^{ss}, and, correspondingly, there is an open and dense stratum $\mathcal{S}_{(H)}$ in $M//G$. Let us denote the open dense piece $F_\infty^{-1}(Z_{(H)})$ as $F_\infty^{-1}(Z^O)$, and denote the corresponding open and dense stratum $\mathcal{S}_{(H)}$ of $M//G$ as $\mathcal{S}^O$.

Definition 4. Let $s_1, s_2 \in \mathcal{H}(M, L^{\otimes k})^G$ and let $r_1, r_2 \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. We define

$$\langle s_1, s_2\rangle_{(1)} = \int_M^{(1)} (s_1, s_2)\mathrm{dvol}(M) = \int_{F_\infty^{-1}(Z^O)} (s_1, s_2)\mathrm{dvol}(F_\infty^{-1}(Z^O)),$$

and we define

$$\langle r_1, r_2\rangle_{(1)} = \int_M^{(1)} (r_1, r_2)\mathrm{dvol}(M) = \int_{F_\infty^{-1}(Z^O)} (r_1, r_2)\mathrm{dvol}(F_\infty^{-1}(Z^O)).$$

By Theorems 11 and 12, we have

Corollary 2. *Let $s \in \mathcal{H}(M, L^{\otimes k})^G$, and let $r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. Then,*

$$\|s\|_{(1)}^2 = \int_M^{(1)} |s|^2\mathrm{dvol}(M) = (k/2\pi)^{d_{\mathcal{S}^O}/2} \int_{\mathcal{S}^O} |A_k' s|^2([x]) I_k^{\mathcal{S}^O}([x])\epsilon_{\hat{\omega}_{\mathcal{S}^O}} + II_k^{Z^O},$$

where, $I_k^{\mathcal{S}^O}([x]) = 1$ or $\lim_{k\to\infty} I_k^{\mathcal{S}^O}([x]) = 2^{-d_{G/H}/2}\mathrm{vol}(G \cdot x)$ uniformly for $[x] \in \mathcal{S}^O$ for some $H \neq G$, and, $II_k^{Z^O} = 0$ or $\lim_{k\to\infty} II_k^{Z^O} = 0$;

$$\|r\|_{(1)}^2 = \int_M^{(1)} |r|^2\mathrm{dvol}(M) = (k/2\pi)^{d_{\mathcal{S}^O}/2} \int_{\mathcal{S}^O} |B_k' r|^2([x]) J_k^{\mathcal{S}^O}([x])\epsilon_{\hat{\omega}_{\mathcal{S}^O}} + \widetilde{II}_k^{Z^O},$$

where, $J_k^{\mathcal{S}^O}([x]) = 1$ or $\lim_{k\to\infty} J_k^{\mathcal{S}^O}([x]) = 1$ uniformly for $[x] \in \mathcal{S}^O$, and, $\widetilde{II}_k^{Z^O} = 0$ or $\lim_{k\to\infty} \widetilde{II}_k^{Z^O} = 0$.

The following definition modifies the usual definition of quantum norms, but it takes into account all the strata. Physical interpretations of this definition would be desirable.

Definition 5. Let $s_1, s_2 \in \mathcal{H}(M, L^{\otimes k})^G$ and let $r_1, r_2 \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. We define

$$\langle s_1, s_2\rangle_{(2)} = \int_M^{(2)} (s_1, s_2)\mathrm{dvol}(M) = \sum_{Z_{(H)}} \int_{F_\infty^{-1}(Z_{(H)})} (s_1, s_2)\mathrm{dvol}(F_\infty^{-1}(Z_{(H)})),$$

and we define

$$\langle r_1, r_2\rangle_{(2)} = \int_M^{(2)} (r_1, r_2)\mathrm{dvol}(M) = \sum_{Z_{(H)}} \int_{F_\infty^{-1}(Z_{(H)})} (r_1, r_2)\mathrm{dvol}(F_\infty^{-1}(Z_{(H)})).$$

Again, by Theorems 11 and 12, we have

Corollary 3. *Let $s \in \mathcal{H}(M, L^{\otimes k})^G$, and let $r \in \mathcal{H}(M, L^{\otimes k} \otimes \sqrt{K})^G$. Then,*

$$\|s\|_{(2)}^2 = \int_M^{(2)} |s|^2\mathrm{dvol}(M)$$
$$= \sum_{\mathcal{S}_{(H)}} (k/2\pi)^{d_{\mathcal{S}_{(H)}}/2} \int_{\mathcal{S}_{(H)}} |A_k' s|^2([x]) I_k^{\mathcal{S}_{(H)}}([x])\epsilon_{\hat{\omega}_{\mathcal{S}_{(H)}}} + \sum_{Z_{(H)}} II_k^{Z_{(H)}},$$

where, $I_k^{\mathcal{S}_{(G)}} = 1$ or $\lim_{k\to\infty} I_k^{\mathcal{S}_{(H)}}([x]) = 2^{-d_{G/H}/2}\mathrm{vol}(G\cdot x)$ uniformly for $[x] \in \mathcal{S}_{(H)}$ with $H \neq G$, and, $II_k^{Z_{(H)}} = 0$ or $\lim_{k\to\infty} II_k^{Z_{(H)}} = 0$;

$$\|r\|_{(2)}^2 = \int_M^{(2)} |r|^2\mathrm{dvol}(M)$$
$$= \sum_{\mathcal{S}_{(H)}} (k/2\pi)^{d_{\mathcal{S}_{(H)}}/2} \int_{\mathcal{S}_{(H)}} |B_k' r|^2([x]) J_k^{\mathcal{S}_{(H)}}([x])\epsilon_{\hat{\omega}_{\mathcal{S}_{(H)}}} + \sum_{Z_{(H)}} \widetilde{II}_k^{Z_{(H)}},$$

where, $J_k^{\mathcal{S}(G)} = 1$ *or* $\lim_{k\to\infty} J_k^{\mathcal{S}(H)}([x]) = 1$ *uniformly for* $[x] \in \mathcal{S}_{(H)}$, *and,* $\widetilde{II}_k^{Z(H)} = 0$ *or* $\lim_{k\to\infty} \widetilde{II}_k^{Z(H)} = 0$.

For both Definitions 4 and 5, we have the following asymptotic unitarity for the maps B'_k.

Theorem 14. *The maps* B'_k *are asymptotically unitary, in the sense that*

$$\lim_{k\to\infty} \|B_k'^{*} B'_k - I\| = \lim_{k\to\infty} \|B'_k B_k'^{*} - I\| = 0,$$

where $\|.\|$ *refers to the operator norm.*

Proof. We use Theorem 9, the definitions in (3) and in (4) of Section 5, and we use the above results in Corollaries 2 and 3 of Theorems 11 and 12. For the case of the quantum norms in Definition 5, we also use the fact that there are finitely many strata. One may refer to [9], for the proof of Theorem 5.2 for the asymptotic unitarity of the maps B_k. □

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